UNIVERSITY OF GLOBAL VILLAGE (UGV),BARISHAL

Differential and Integral Calculus

MAT 0541-1101
Course Content

$$\frac{dy}{dx}$$

$$\int_a^b f(x)\,dx$$

Prepared By:
G.M Zahidul Islam
Lecturer
University Of Global Village (UGV),Barishal

BASIC COURSE INFORMATION

Course Code:	MAT 0541-1201
Course Code:	MAT 0541-1101
Credits:	3
Course Type	GEd
CIE Marks	90
SEE Marks	60
Exam Hours	03

Course Assignment Pattern

Quiz

Altogether 4 quizzes may be taken during the semester, 2 quizzes will be taken for midterm and 2 quizzes will be taken for final term

Assignment

Altogether 4 assignments may be taken during the semester, 2 assignments will be taken for midterm and 2 assignments will be taken for final term.

Presentation

The students will have to form a group of maximum 3 members. The topic of the presentation will be given to each group and students will have to do the group presentation on the given topic

Assignment Pattern

CIE – Continuous Internal Evaluation (90 Marks)

Bloom's Category Marks (out of 90)	Mid Exam (45)	Assignment (15)	Quiz (15)	Attendance & External Participation in Curricular/Co-Curricular Activities (15)
Remember	5		05	
Understand	5	05		
Apply	10		05	15
Analyze	10		05	
Evaluate	10			
Create	5	10		

Assignment Pattern

SEE – Semester End Examination (60 Marks)

Bloom's Category	Final Examination
Remember	10
Understand	10
Apply	15
Analyze	10
Evaluate	10
Create	05

Course Learning Outcomes (CLO'S)

CLO 1
To describe the basic concepts of limits, derivatives, and integrals.

CLO 2
Recognize the appropriate tools of calculus to limits, derivatives, and integrals.

CLO 3
Categorize problems and correlate the problems with well-known methods.

CLO 4
Analyze and evaluate some real life/applied problems.

CLO 5
Modify the previous result/ Develop a new methodology.

Course Rationales

The course is based upon two concepts, namely, the concept of function and the concept of limit. Differential Calculus helps to find the rate of change of quantity. It is mainly focused on some important topics such as limits, continuity and differentiability. The rules of differentiation are introduced, and methods of differentiating various algebraic and transcendental functions will be developed. Applications of differential calculus to finding maximum, minimum, tangent, and normal values of the function. Integral calculus is the reciprocal of differentiation. Methods of algebraic integration will be introduced, with both definite and indefinite integrals being determined for a variety of functions. It is generally used for calculating areas. At the end of each chapter, the assignment is compulsory to make students involve using knowledge of this course to implement and solve different kinds of problems

Course Objective

The objective of this course is to

- Teach basic math skills in calculus and make students familiar with Limits, Continuity, Derivatives, Curves, and Integrals.
- Demonstrate the solutions of the problems in the field of Engineering using calculus. Thus, the students can proceed with their studies towards advanced courses in their fields.

Course Summary

Sl.	Content of Course	Hrs	CLOs
01	**Functions and their Graphs, Limit, continuity.** ➢ Definitions of functions. ➢ Kinds of functions (even, odds, inverse, one-one, onto etc.) ➢ Graph of some well-known functions. ➢ Defines continuity at a point using limits. ➢ Use rules of limits. ➢ Evaluate limits by way of tables and graphs. ➢ Determine the existence of and find limits at real numbers. ➢ Evaluate limits algebraically by means of substitution, factoring, and using special limits. ➢ Use limits to determine whether a function is continuous at a point	**12**	**CLO1**
02	**Differentiability and its applications** ➢ Express the derivative of a function as a limit. ➢ Use formulas to take derivatives of polynomial, radical, exponential, and logarithmic functions. ➢ Relate the first derivative to velocity and the second derivative to acceleration. ➢ Use the product and quotient rules to take derivatives. ➢ Solve applied problems involving derivatives	08	**CLO 1** **CLO 2**

Course Summary

Sl.	Content of Course	Hrs	CLOs
03	**Maxima & Minima, Rolle's, and Mean Values Theorem** ➢ Find critical numbers and critical points. ➢ Find intervals where a function is increasing or decreasing. ➢ Find absolute extrema on a closed interval. ➢ Find relative extrema using the first derivative test ➢ Solve application problems. ➢ To find the maximum or minimum value of a particular quantity. Such applications exist in economics, business, and engineering	**10**	**CLO 3**
04	**Integration and methods of Integrations** ➢ Definitions of integrations ➢ Important formula of integrations ➢ Types of integrations (Definite and Indefinite) ➢ Fundamental theorem of integrations ➢ Evaluating definite integral by substitutions ➢ Integrations by partial fractions	08	**CLO 4**

Course Summary

Sl.	Content of Course	Hrs	CLOs
05	**Gamma Function and Beta Function** ➢ Definitions of Gamma functions ➢ Some basic properties of gamma functions ➢ Applications of gamma functions ➢ Definitions of Beta functions ➢ Some basic properties of beta functions ➢ Applications of beta functions	10	CLO 4 CLO 5
06	**Applications of integrations** **(Area and Volumes related problems)** ➢ Evaluate definite integrals to find the net area between a curve and the x-axis using the Fundamental Theorem of Calculus. ➢ Use basic integration properties to solve graphical net area problems. ➢ Use properties to definite integrals to solve graphical net area problems. ➢ Use definite integrals to find the area between two curves	12	CLO 4 CLO 5

Course Plan Mapped With CLO

Week No.	Topics	Teaching Strategy	Assessment Strategy	Corresponding CLO
01	Basic concept of function, Domain, Co-domain, Range	Lecture, Multimedia	Feedback, Q&A	CLO1
02	Kinds of function(even, odds, one-one, onto, bijective etc.)	Lecture, Multimedia	Feedback, Q&A	CLO1
03	Inverse function and Composite function, Graph of some well-known functions	Lecture, Multimedia	Feedback, Q&A	CLO1
04	Basic concept of limit, Evaluate limit	Lecture, Multimedia	Feedback, Q&A	CLO1
05	Existence of limit and Continuity of a function at a point.	Lecture, Multimedia	Feedback, Q&A	CLO1

Course Plan Mapped With CLO

Week No.	Topics	Teaching Strategy	Assessment Strategy	Corresponding CLO
06	Basic concept of differentiation, differentiability, Formulae of differentiation, Differentiation (Sum & Difference Rules, Product Rule & Quotient Rule)	Lecture, Multimedia	Feedback, Q&A	CLO1
07	Differentiation (The Chain Rule, Function as Power of another Function, Differentiation Parametric Equations, Implicit Function)	Lecture, Multimedia	Feedback, Q&A	CLO2
08	Successive differentiation, Leibnitz's theorem	Lecture, Multimedia	Feedback, Q&A	CLO2
09	Solve applied problems involving derivatives, Relate the first derivative to velocity and the second derivative to acceleration, Find critical numbers and critical points, Find intervals where a function is	Lecture, Multimedia	Feedback, Q&A	CLO1 CLO2

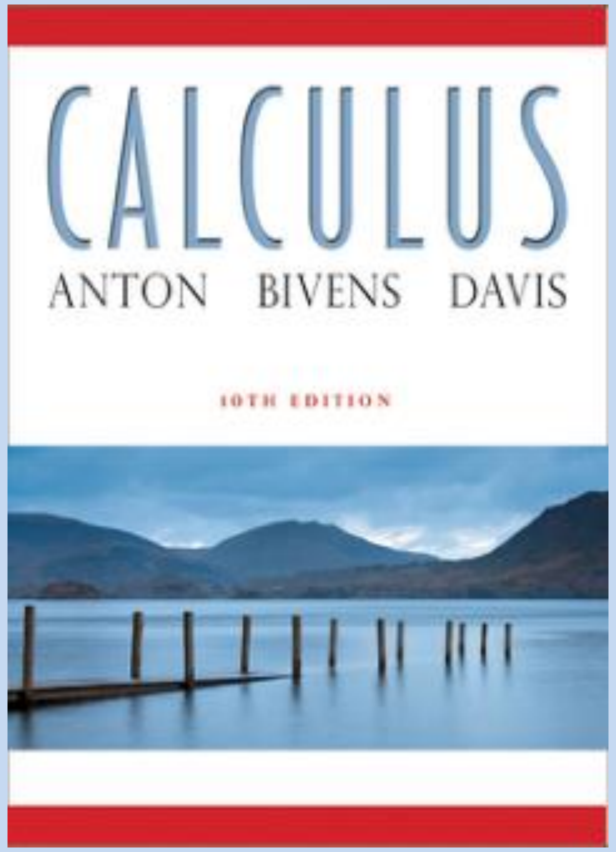
Course Plan Mapped With CLO

Week No.	Topics	Teaching Strategy	Assessment Strategy	Corresponding CLO
10	To find the maximum or minimum value of a particular quantity. Such applications exist in economics, business, and engineering	Lecture, Multimedia	Feedback, Q&A	CLO3
11	Rolle’s, and Mean Values Theorem	Lecture, Multimedia	Feedback, Q&A	CLO3
12	Definitions of integrations, Important formula of integrations, Types of integrations (Definite and Indefinite) Fundamental theorem of integrations	Lecture, Multimedia	Feedback, Q&A	CLO4
13	Evaluating definite integral by substitutions, Some Ideal Integrals, Type based integration.	Lecture, Multimedia	Feedback, Q&A	CLO4

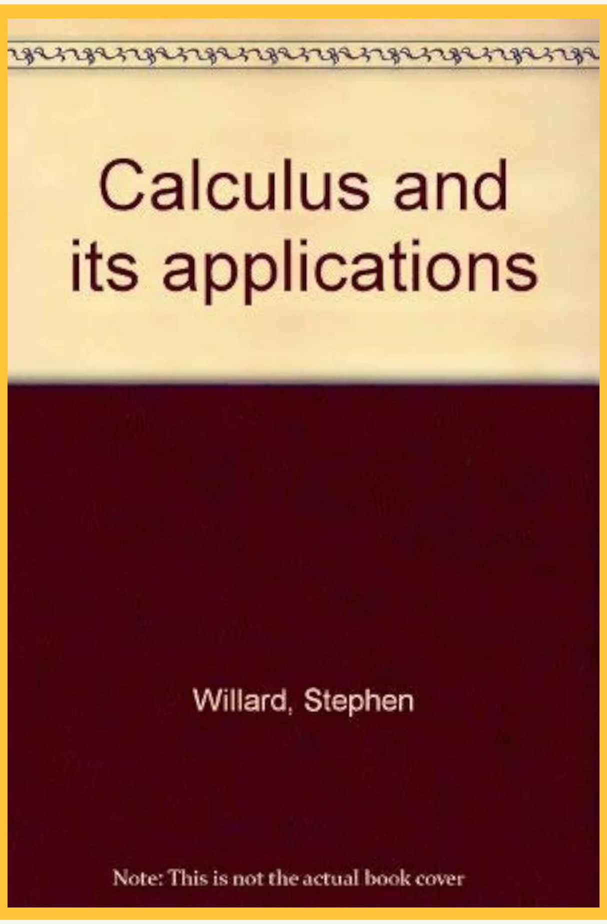
Course Plan Mapped With CLO

Week No.	Topics	Teaching Strategy	Assessment Strategy	Corresponding CLO
14	Integrations by parts, Definite integral's special properties and exercise	Lecture, Multimedia	Feedback, Q&A	CLO4
15	Definitions of Gamma functions and Beta functions, Some basic properties of gamma functions and Beta function, Applications of gamma functions and Beta functions .	Lecture, Multimedia	Feedback, Q&A	CLO4
16	Evaluate definite integrals to find the area between a curve and the x or y-axis, Use definite integrals to find the area between two curves.	Lecture, Multimedia	Feedback, Q&A	CLO4 CLO5
17	Area under a plane curve in polar co-ordinates	Lecture, Multimedia	Feedback, Q&A	CLO4 CLO5

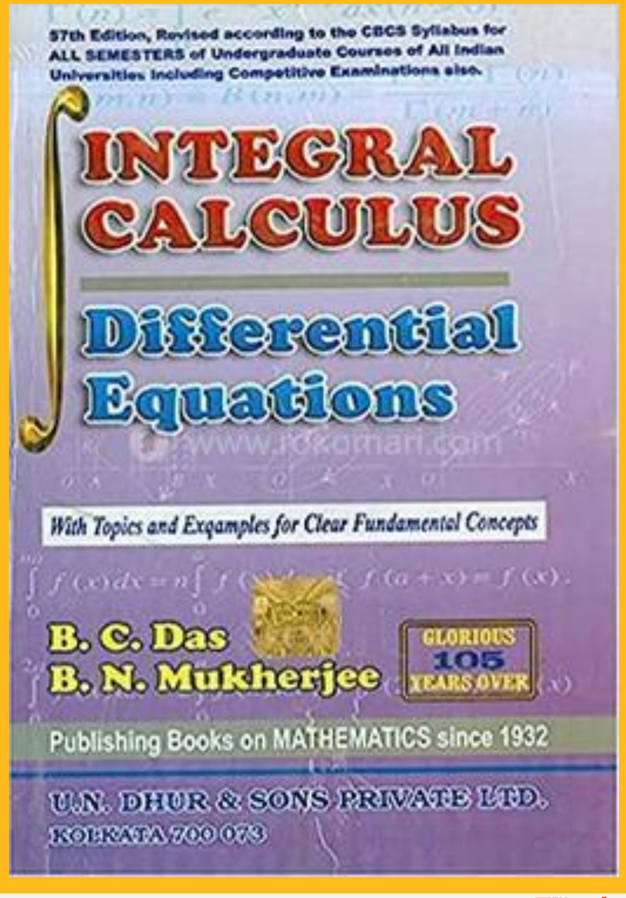
References



Calculus, 10th edition, Howard Anton, Irl Bivens, Stephens Davis.



Wilard Stephen: Calculus and its application



Das& Mukherjee: Differential and Integral Calculus

WEAK 1

Page:17-25

FUNCTION

Definition:

A relation from a set A to a set B is called a function if

(i) Each element of set 'A' is associated with some element in set 'B'.

(ii) Each element of set 'A' has unique image in set 'B'.

Example.

$\therefore f \equiv \{(1, a), (2, b), (3, c)\}$

So, it can be said that $f \subset A \times B$.

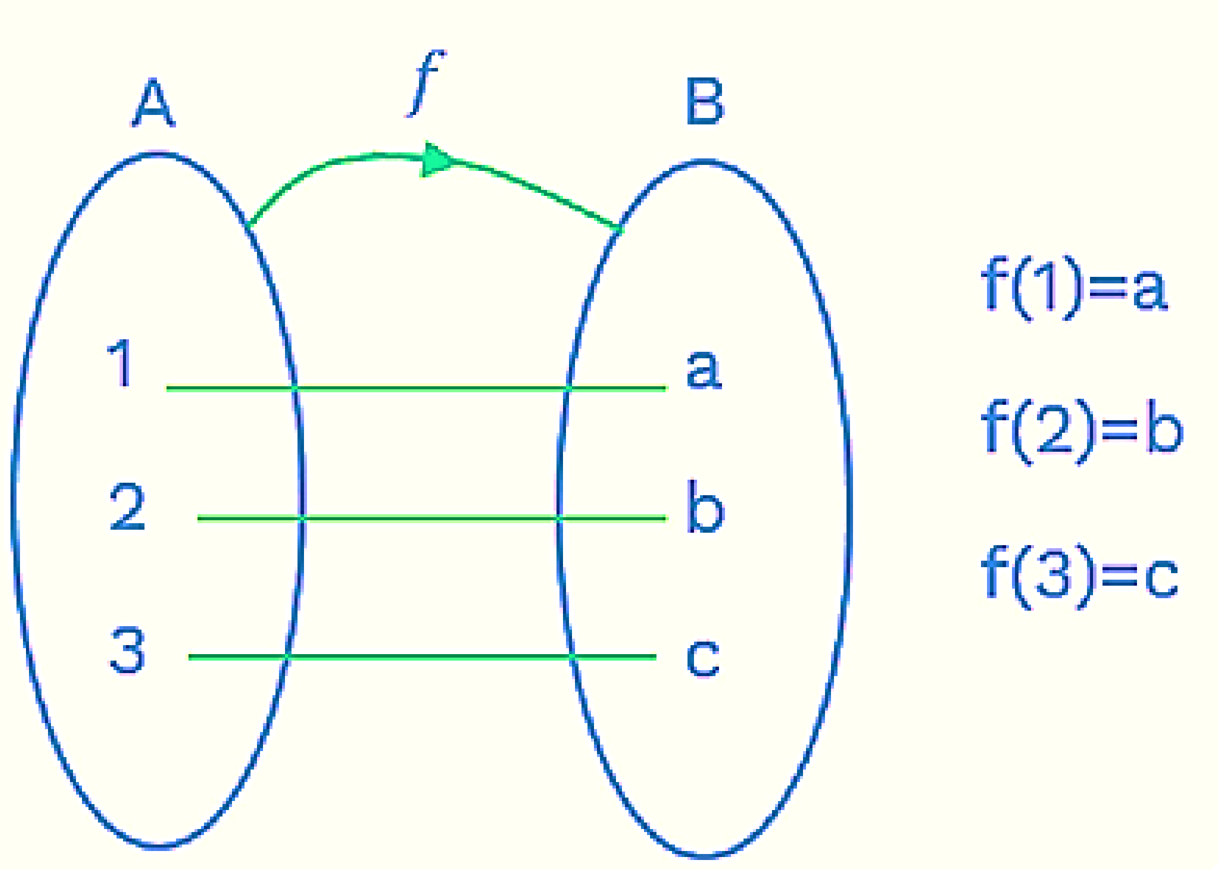


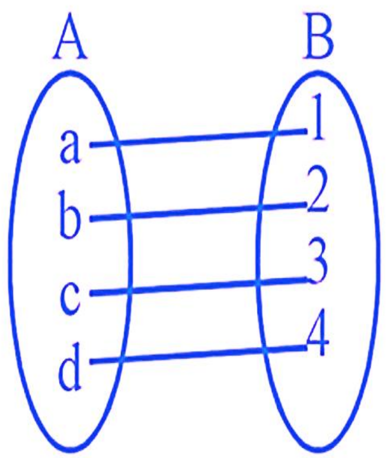
Image and Pre-image

If an element $(a \in A)$ is associated with an element $(1 \in B)$, then '1' is called, the

"f image of a" or "image of a under f"

or

"the value of the function f at a"

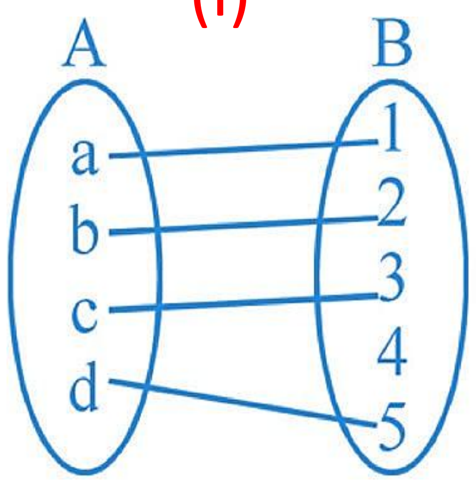


"a is called the pre-image of 1"

Example. A = {a, b, c, d}, B = {1, 2, 3, 4, 5}

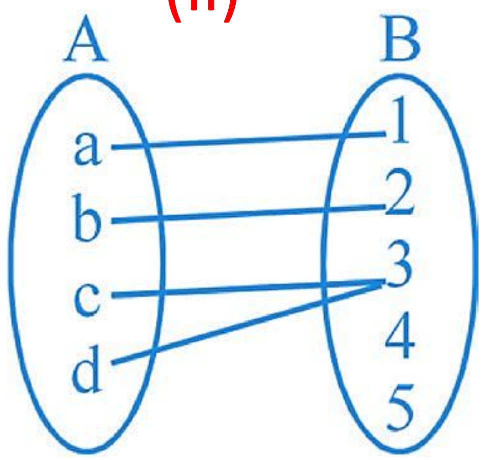
f: $A \to B$

(i)



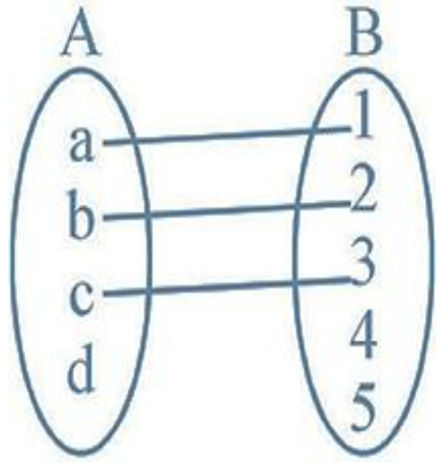
is a function.
Every element in A has a unique image in B

(ii)



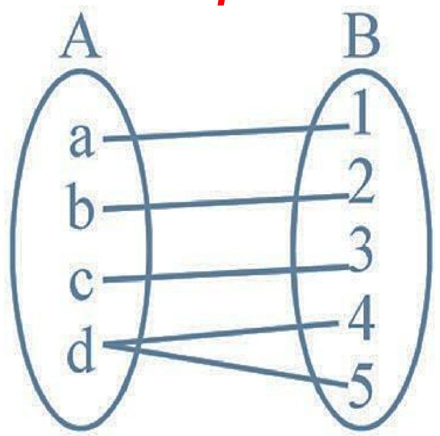
is a function.
Every element in A has a unique image in B

(iii)



is not a function.
('d' has no image in B)

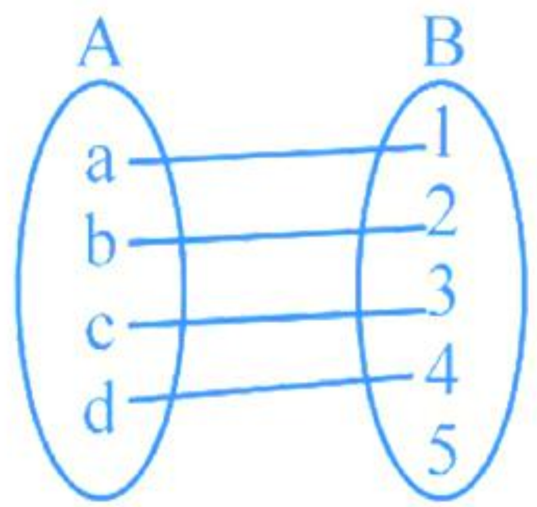
(iv
)



is not a function.
('d' does not have a unique image in B)

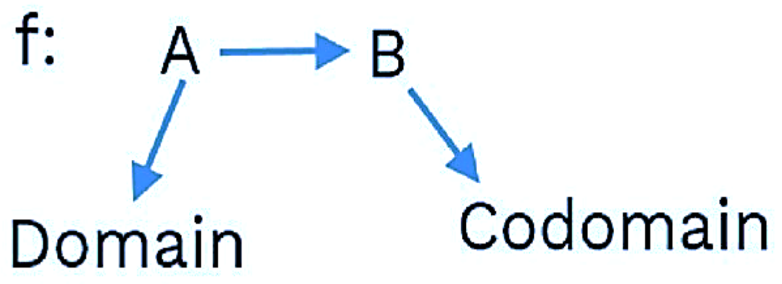
Domain, Codomain and Range

A = {a, b, c, d}, B = {1, 2, 3, 4, 5}



Domain	→ {a, b, c, d}
Co-domain	→ {1, 2, 3, 4, 5}
Range	→ {1, 2, 3, 4}

Note: Range ⊆ Co-domain



Note:

When only "rule of function" is given

(i) It is called "Real valued function".

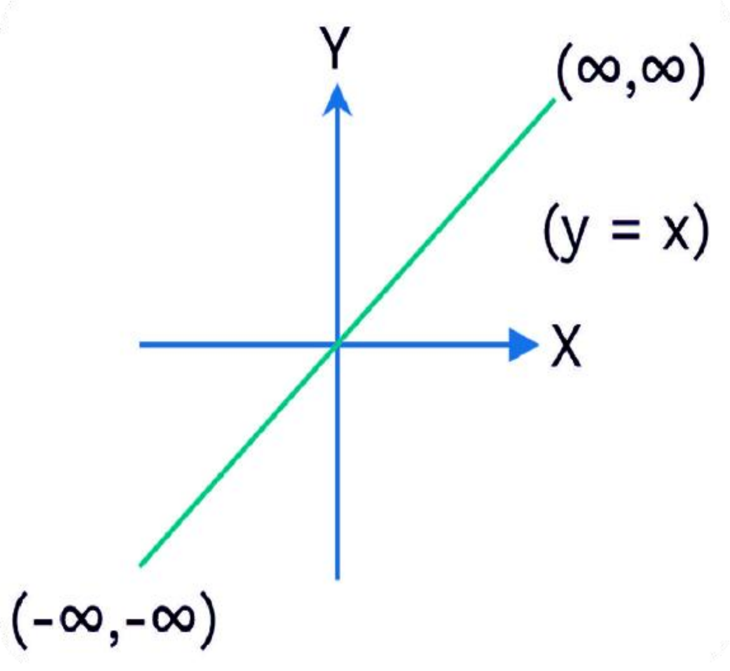
(ii) Domain =Set of real 'x' for which y is real (Input values).

(iii)Range =Set of all real y values obtained after putting real x in domain (All output values).

Range can be said to be the collection of functional outputs.

Question: **Find domain of**

(i) $y = x$



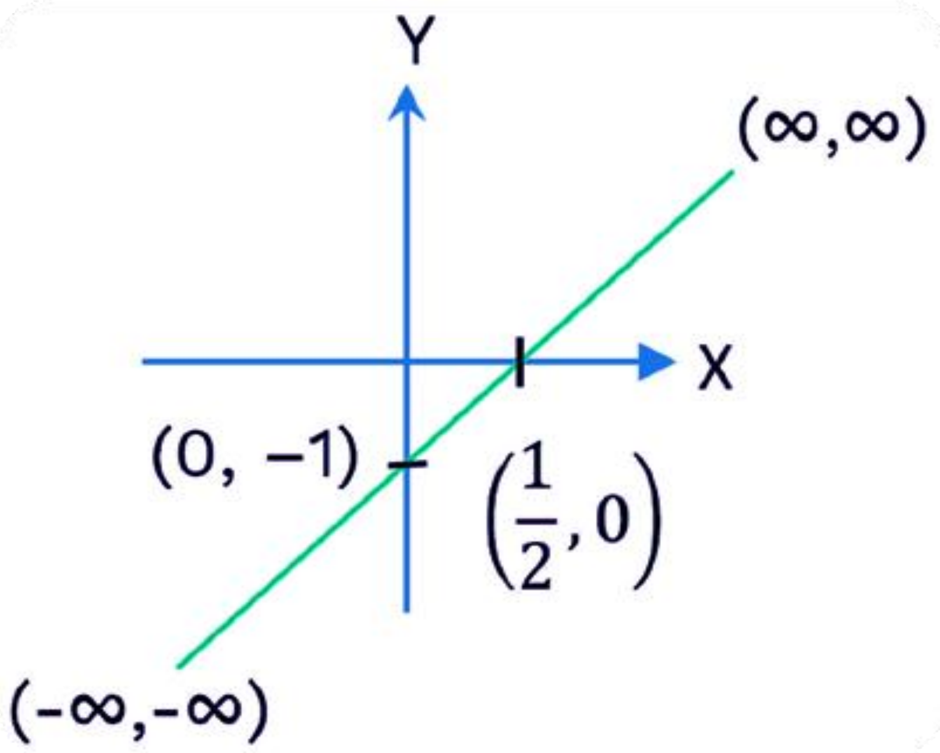
Solution:

Here, it can be seen that all the values in $(-\infty, \infty)$ can be used as input as we move from left to right of graph.

At the same time, y achieves all values in $(-\infty, \infty)$ as we move from bottom to top of graph.

$\therefore$ Domain $\equiv x \in R$ & Range $\equiv y \in R$

(ii) $y = 2x - 1$



Solution:

Domain $\equiv x \in R$

For range, it can be seen from graph

Range $\equiv y \in R$

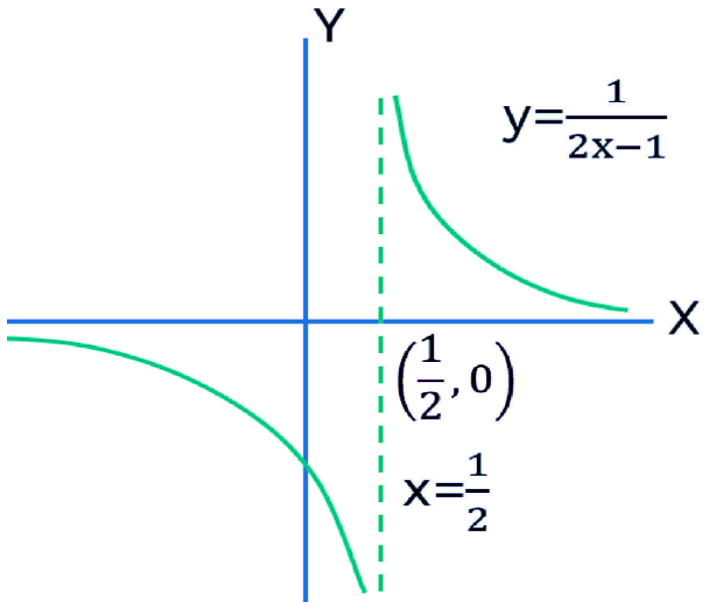
(iii) $y = \frac{1}{2x-1}$

Solution:

Domain:

Here, $2x - 1 \neq 0$ or , $x \neq \frac{1}{2}$

$\therefore$ Domain $\equiv \mathbf{x} \in \mathbf{R} - \{\frac{1}{2}\}$



Range:

$y = \frac{1}{2x-1}$

or, $2yx - y = 1$

or, $x = \frac{1+y}{2y}$

Here, $y \neq 0$

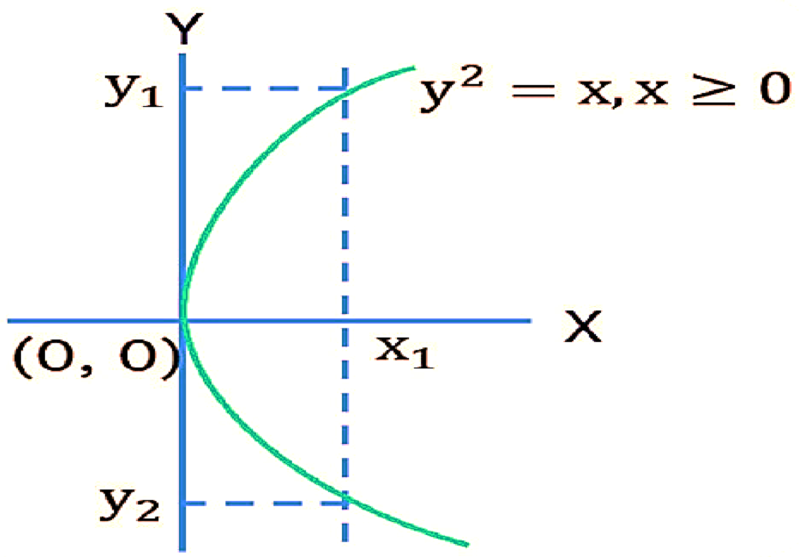
$\therefore$Range $\equiv \mathbf{x} \in \mathbf{R} - \{\frac{1}{2}\}$

Point to Remember!!!

If any vertical line cuts a curve at at least two different points, then the curve cannot be a function.

Point to Remember!!!

$y^2 = x$ is curve not function.
For $x \geq 0 \Rightarrow y = \pm\sqrt{x}$ So, for same value of x, there is two values of y. So, it cannot be a function.



Is not a function

(iv) $y = \sqrt{2x - 1}$

Solution:

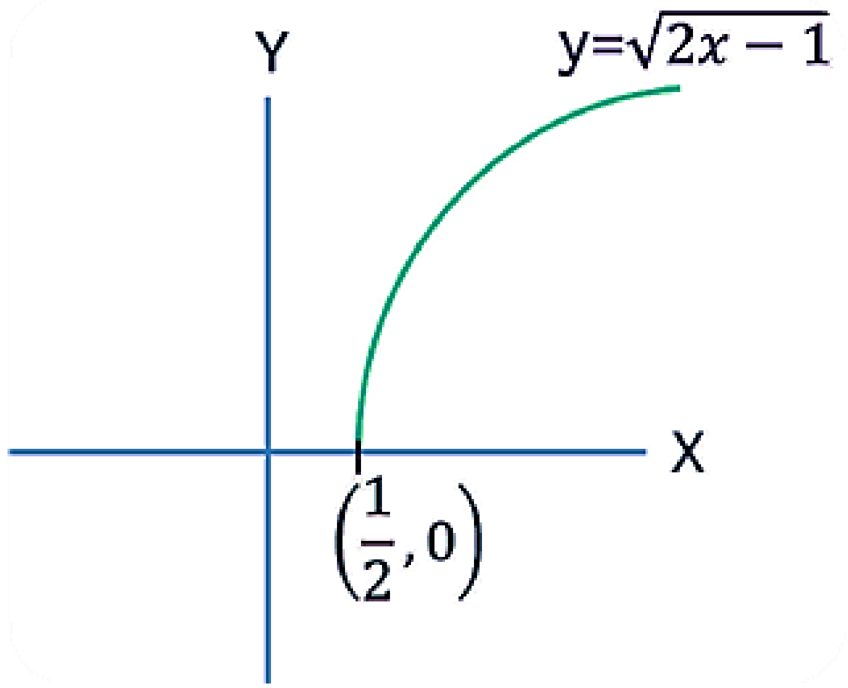
Domain: here, $2x - 1 \geq 0$ or, $x \geq \frac{1}{2}$

$\therefore$ Domain $\equiv \mathbf{x} \in [\frac{1}{2}, \infty)$

Range:

If $x = \frac{1}{2}$ then $y = 0$ and if $x \to \infty$ then $y \to \infty$

$\therefore$Range $\equiv \mathbf{y} \in (0, \infty)$

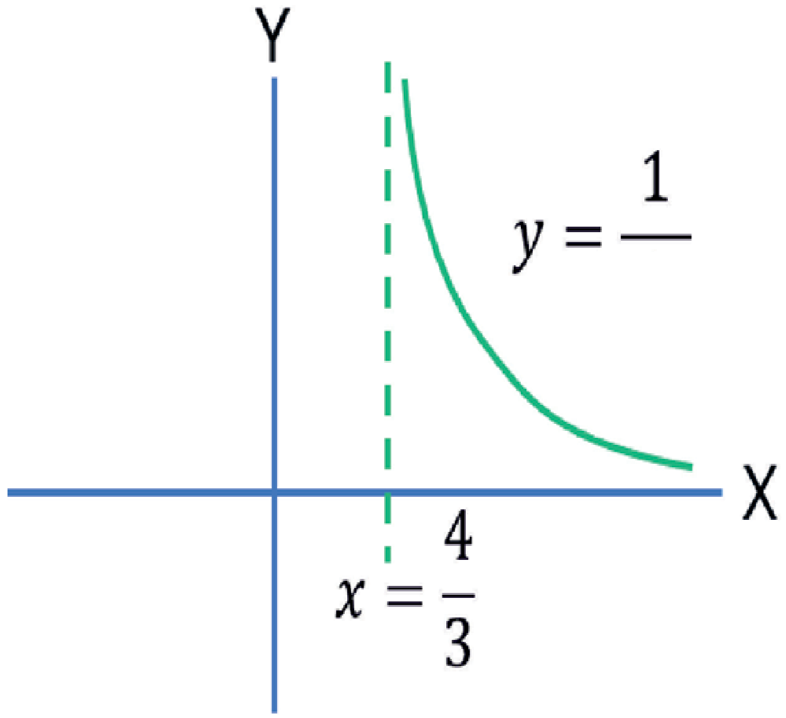


(iv) $y = \frac{1}{\sqrt{3x-4}}$

Solution:

Domain $\equiv \mathbf{x} \in \left(\frac{4}{3}, \infty\right)$

Range $\equiv \mathbf{y} \in (0, \infty)$



Question: $f(x) = \sqrt{x^2 + ax + 4}$

(a) Find 'a' if range is [2, ∞).

(b) Find 'a' if domain is all real.

Solution:

(a) Since, $\sqrt{x^2 + ax + 4} \in [2, \infty)$.

or, $x^2 + ax + 4 \in [4, \infty)$.

∴ Minimum value of $x^2 + ax + 4 = 4$

$\Rightarrow \frac{-D}{4a} = 4$ Here, D=$b^2 - 4ac$

$\Rightarrow \frac{16 - a^2}{4} = 4$

$\Rightarrow a = 0$

(b) Domain is all real.

It means $x^2 + ax + 4 \geq 0 \ \forall \ x \in R$.

∴ $D \leq 0$ Here, D=$b^2 - 4ac$

$a^2 - 16 \leq 0$

$a \in [-4, 4]$

Exercise

(i) $y = \frac{3x-4}{2x-1}$

(ii) $y = \frac{1}{\sqrt{2x-1}}$

(Iii) $y = \sqrt{4 - x^2}$

(iv) $y = \sqrt{x^2 - 4}$

(v) $y = \ln(2x + 3)$

WEAK
2

Page:17-25

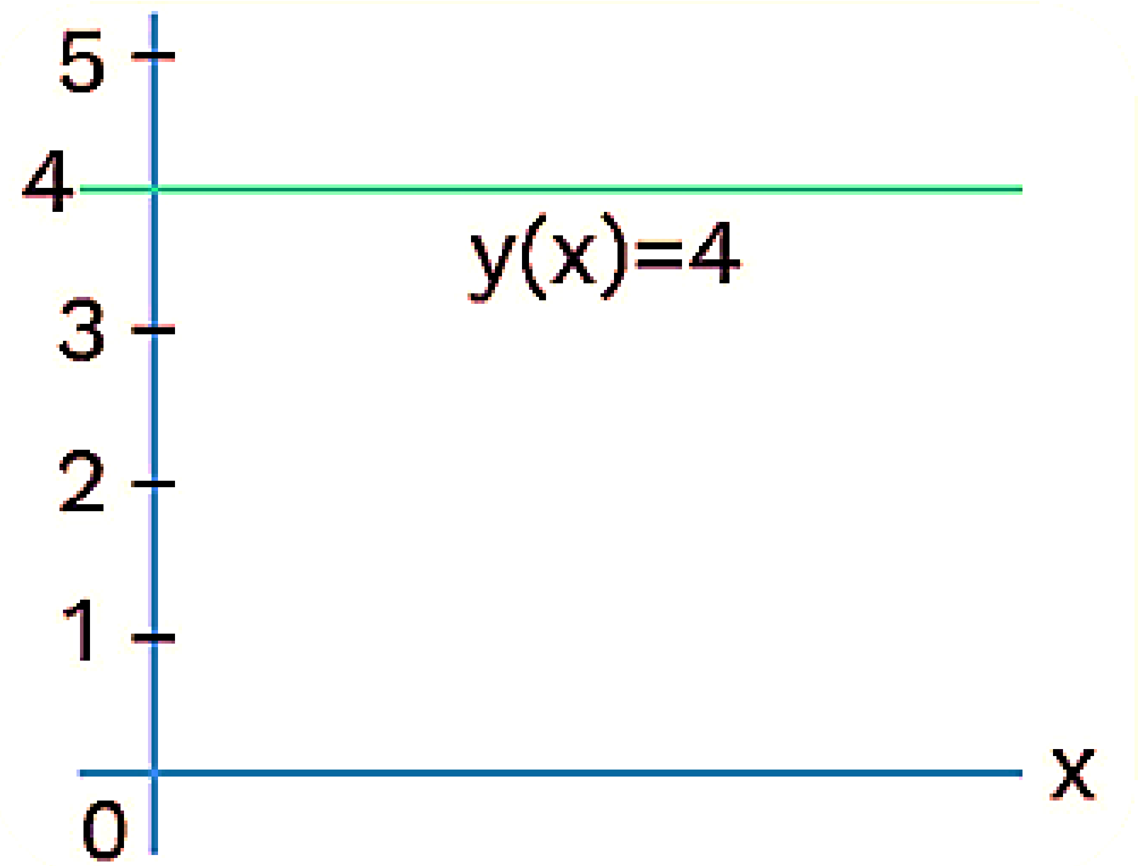
Definition of Constant function:

Constant function is a function whose (output) value is the same for every input value.

Example:

For example, the function given is a constant function because the value is 4 regardless of the input value(see diagram)

In this type of function, domain is (-∞, ∞) while range contains only a single value. In above example range is {4}.



Definition of identity function:

An identity function, also called an identity relation or identity map or identity transformation, is a function that always returns the same value that was used as its argument.

Example:

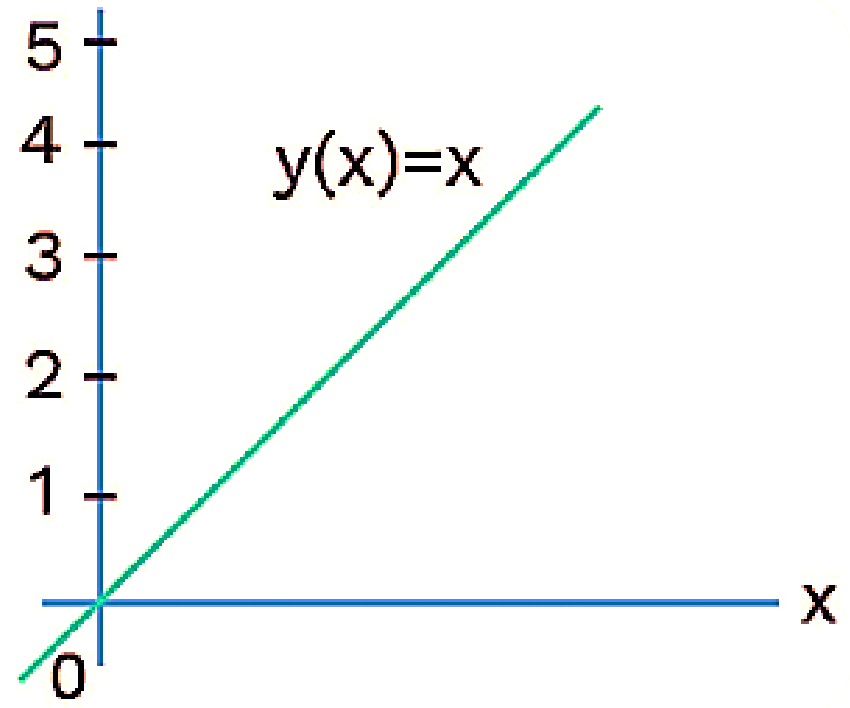
$f(x) = x$

Domain = R

Range is $(-\infty, \infty)$

This is an increasing function.

It is also represented by I_x.



Definition of odd and even functions:

A function f(x) defined on the symmetric interval (-a, a)

If f(-x) = f(x) for all x in the domain of 'f' then f is said to be an even function.

If f(-x) = -f(x) for all x in the domain of 'f' then f is said to be an odd function.

Example:

(a)

$f(x) = x^2 + x$

$\Rightarrow f(-x) = x^2 - x \neq f(x) \text{ or } -f(x)$

$\therefore x^2 + x$ is neither odd nor even.

(b)

$h(x) = \frac{f(x)+f(-x)}{2}$

$\therefore h(-x) = \frac{f(-x)+f(x)}{2} = h(x)$

So, it is even function.

(c)

$h(x) = f(x) - f(-x)$

$\therefore h(-x) = f(-x) - f(x) = -h(x).$

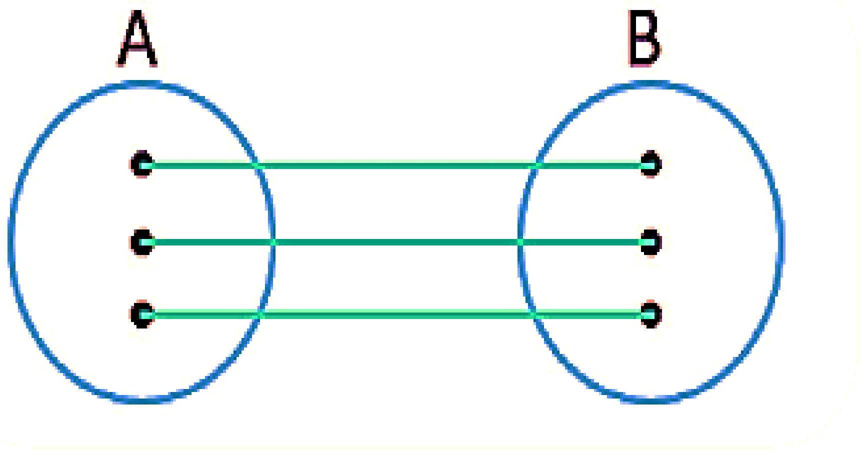
So, it is odd function.

Common odd function: $f = sinx, \ anx, \ x, \ x^3$

Common even function: $f = cosx, \ \sin^2 x, \ x^2, \ |x|, \ |sinx|$

Definition of one-one (injective mapping):

$f: A \rightarrow B$ such that different elements of A have different f images in B.



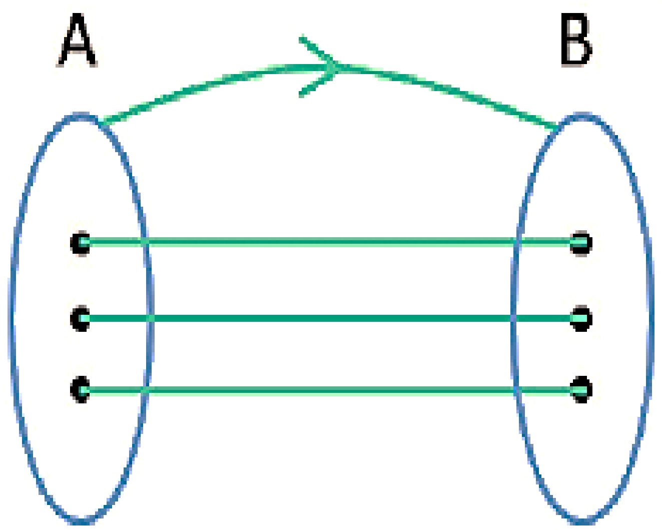
or, $x_1, x_2 \in A$ and $f(x_1), f(x_2) \in B$,

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

or, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

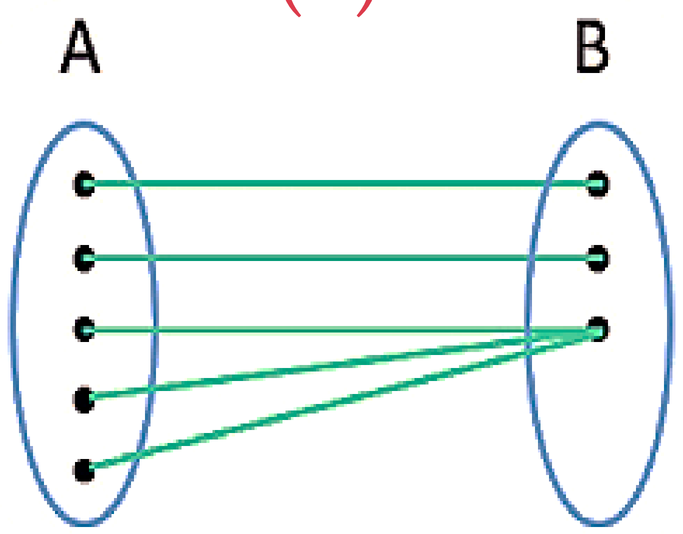
Example:

(i)



one-one (Every input has a different output)

(ii)



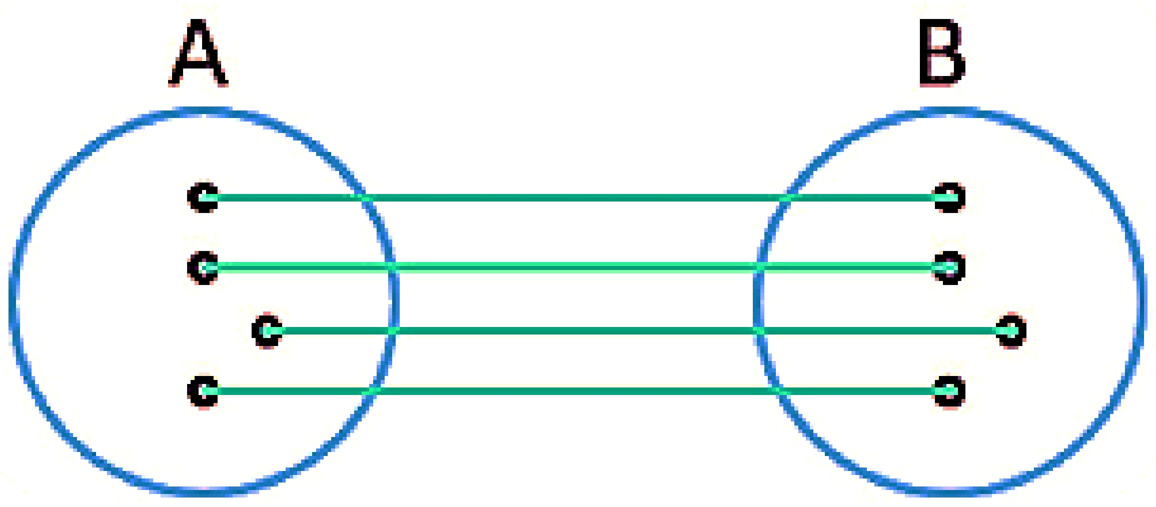
Not one-one (3 inputs have same output)

Definition of onto (surjective mapping):

f: A → B such that each element in B is the f image of at least one element in A

In case of onto function, codomain is equal to range.

So, to find if the function is onto, find range and match it with codomain.



Example:

(i)

$f: R \rightarrow R;\ f(x) = 2x + 1$

Solution:

$y = 2x + 1 \Rightarrow y \in R$

$\Rightarrow$ Codomain = Range

$\Rightarrow$ onto function.

(ii)

$f: R \rightarrow R;\ f(x) = lnx$

Solution:

$y = lnx \Rightarrow y \in R$

$\Rightarrow$ Codomain = Range

$\Rightarrow$ onto function

(iii)

$f: R \rightarrow R^{+};\ f(x) = e^{x}$

Solution:

$y = e^{x} \Rightarrow y \in (0, \infty)$ or R^{+}

$\Rightarrow$ Codomain = Range

$\Rightarrow$ onto function.

Question:

Let the function f: $\mathbf{R} - \{\frac{1}{2}\} \rightarrow \mathbf{R}-\{\frac{1}{2}\}$ is defined by $f(x) = \frac{x-3}{2x-1}$.

Prove that f is an one-one & onto function

Solution:

f is one-one:

Let, $x_1, x_2 \in \mathbf{R} - \{\frac{1}{2}\}$

$\therefore f(x_1) = \frac{x_1-3}{2x-1}$ & $f(x_2) = \frac{x_2-3}{2x_2-1}$

If $f(x_1) = f(x_2)$ then $\frac{x_1-3}{2x_1+1} = \frac{x_2-3}{2x_2+1}$

$\Longrightarrow (x_1 - 3)(2x_2 + 1) = (x_2 - 3)(2x_1 + 1)$

$\Longrightarrow 2x_1x_2 + x_1 - 6x_2 - 3 = 2x_1x_2 + x_2 - 6x_1 - 3$

$\Longrightarrow 7x_1 = 7x_2$

$\Longrightarrow x_1 = x_2$

$\therefore$ f is one -one.

f is onto:

Here $Cod_f = \mathbf{R} - \left\{\frac{1}{2}\right\}$

Let, y= $f(x) = \frac{x-3}{2x+1}$

$\Longrightarrow 2xy + y = x - 3$

$\Longrightarrow 2xy + x = 3 + y$

$\Longrightarrow x = \frac{3+y}{1-2y}$

Here, $1 - 2y \neq 0$

$\Longrightarrow y \neq \frac{1}{2}$

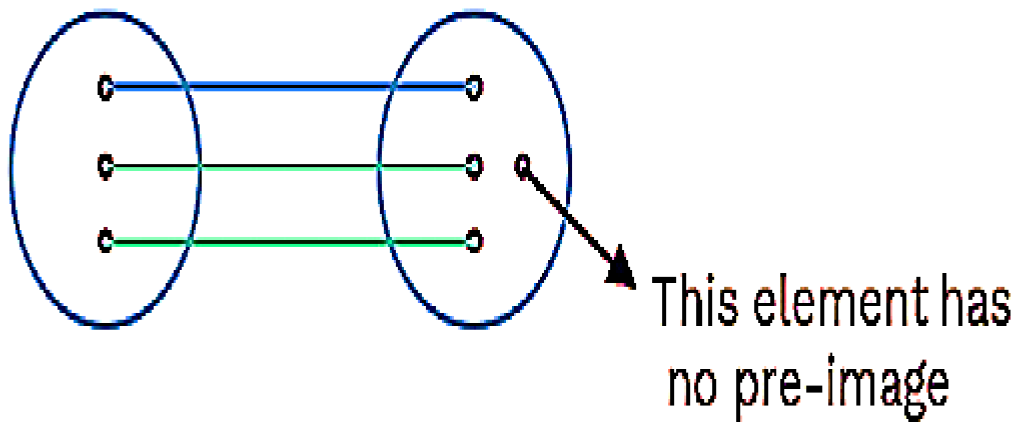
$\therefore R_f$= R- $\frac{1}{2}$

$\therefore Cod_f = R_f$

$\therefore f(x)$ is onto function.

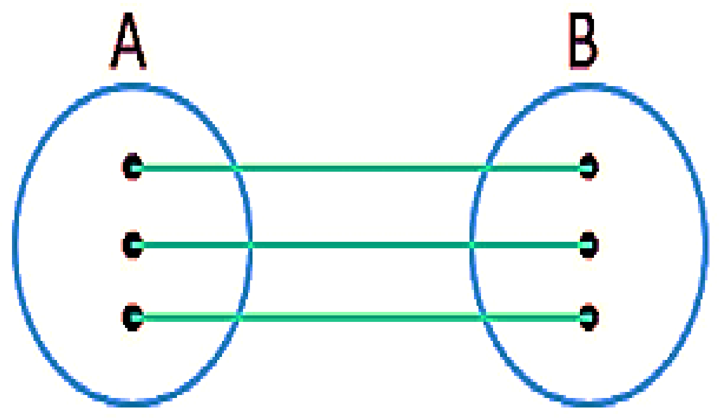
Definition of into function:

f: A → B such that at least on element in B (co-domain) is NOT the f image of any element in domain A.



Bijective Function:

If function is both one-one and onto function, then it is called a bijective function.



Only Bijective functions have inverse functions

Exercise:

(i) Let the function f:R-{3} →R-{1} is defined by $f(x) = \frac{x-2}{x-3}$. Proved that f is one-one and onto function. Find a formula which defined f^{-1}.

(ii) Let, the function $f: R \rightarrow R$ is defined by f(x) = $x^2 + 1$. Proved that f is not one-one & onto function.

WEAK
3

Page:17-25

Inverse function:

Let f be a one-one and onto function with domain A and range B. Then its **inverse function** f^{-1} as domain B and range A and is defined by

$$f^{-1}(y) = x \Leftrightarrow f(x) = y$$

for any y in B.

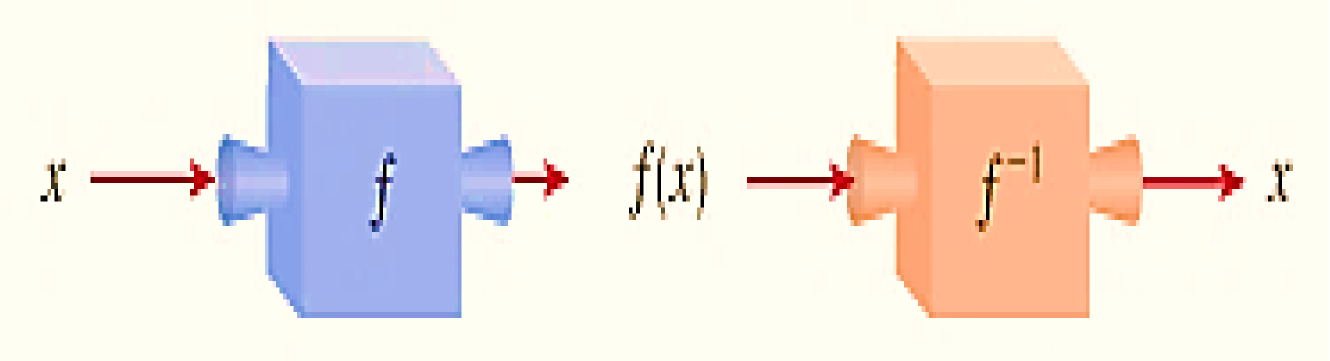
**** Domain of f^{-1} = range of f

range of f^{-1} = domain of f

Caution: Do not mistake the -1 in f^{-1} for an exponent. Thus f^{-1} (x) does not mean $\frac{1}{f(x)}$

$f^{-1}(f(x)) = x$ for every x in A

$f(f^{-1}(x)) = x$ for every x in B



Question:

If f is a one-to-one function and $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$.

Find $f^{-1}(7)$, $f^{-1}(5)$, $f^{-1}(-10)$.

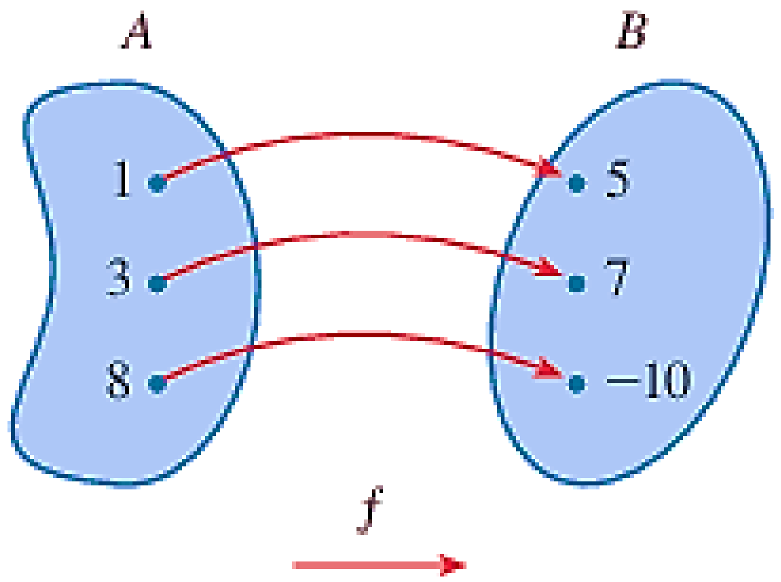
Solution:

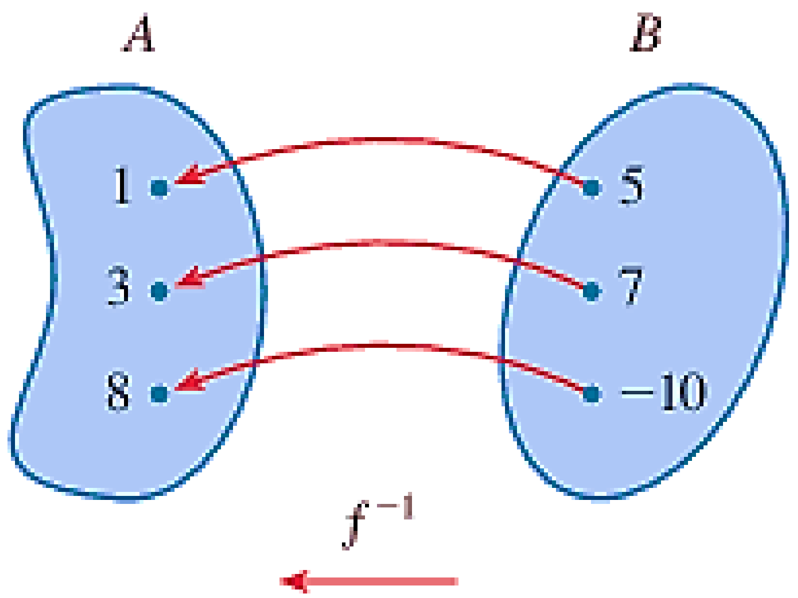
From the definition of f^{-1} we have

$f^{-1}(7) = 3$ because $f(3) = 7$

$f^{-1}(5) = 1$ because $f(1) = 5$

$f^{-1}(-10) = 8$ because $f(8) = -10$





How to Find the Inverse Function of a Bijective Function *f*:

STEP 1: Write $y = f(x)$.

STEP 2: Solve this equation for x in terms of y (if possible).

STEP 3: To express f^{-1} as a function of x, interchange x and y.

The resulting equation is $y = f^{-1}(x)$.

Question:

Find the inverse function of $f(x) = x^3+2$.

Solution:

Let, $y=x^3+2$

Then we solve this equation for x:

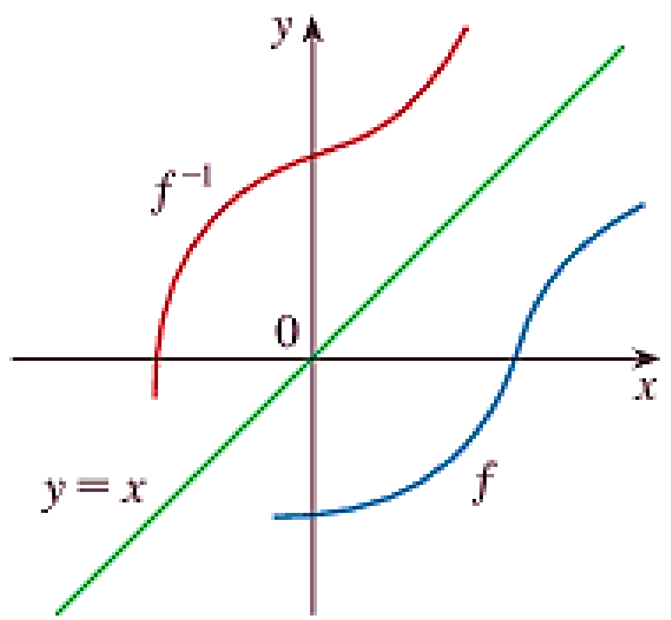
$$x^3 = y - 2$$

$$x = \sqrt[3]{y-2}$$

Finally, we interchange x and y:

$$y = \sqrt[3]{x-2}$$

Therefore the inverse function is $f^{-1}(x) = \sqrt[3]{x-2}$

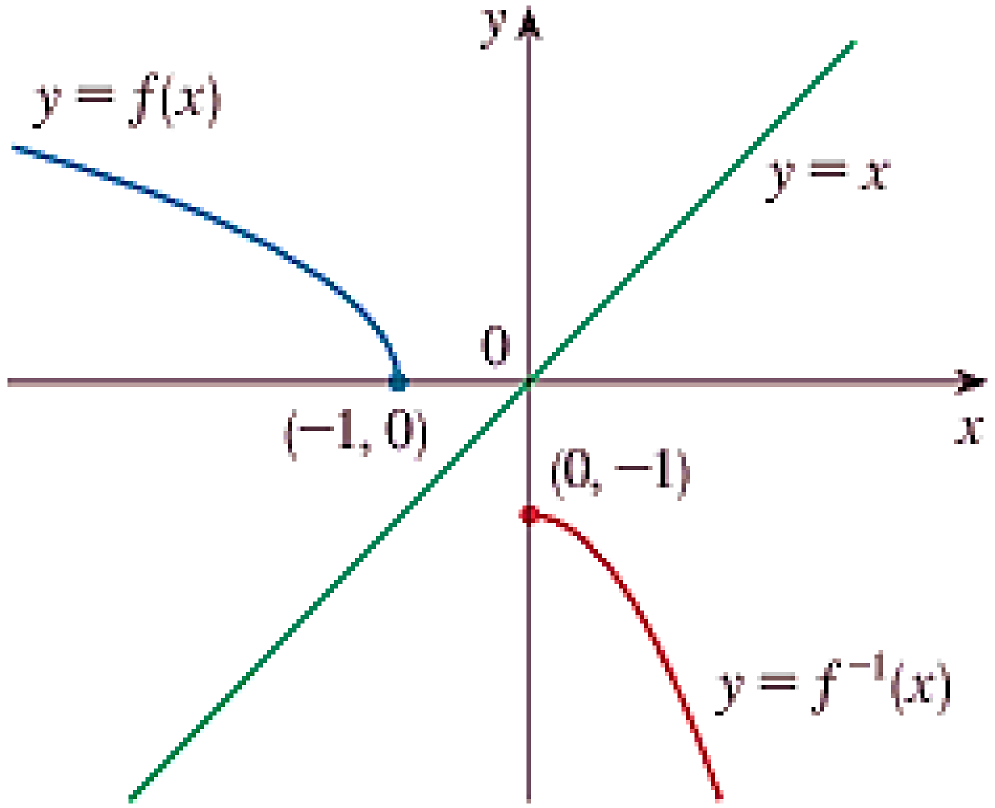


The graph of is f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Question: . Sketch the graphs of $f(x) = \sqrt{-1-x}$ and its inverse function using the same coordinate axes.

Solution:

First we sketch the curve $y = \sqrt{-1-x}$ (the top half of the parabola $y^2 = -1 - x$, or $x = -y^2 - 1$) and then we reflect about the line $y = x$ to get the graph of f^{-1}.



Now, Let $y = \sqrt{-1-x}$

or, $y^2 = -1 - x$

or, $x = -1 - y^2; \quad y \geq 0$

$\therefore f^{-1}(x) = -1 - x^2; \quad x \geq 0$

Question: .

Let $f: R - \left\{-\frac{d}{c}\right\} \to R - \left\{-\frac{d}{c}\right\}$ be a function defined by $f(x) = \frac{ax+b}{cx+d}$.

Find the inverse of $f(x)$

Solution:

Let $y = f(x) = \frac{ax+b}{cx+d}$

Then we get $y = \frac{ax+b}{cx+d}$

$\Rightarrow cxy + dy = ax + b$

$\Rightarrow cxy - ax = -dy + b$

$\Rightarrow x(cy - a) = -dy + b$

$\Rightarrow x = \frac{-dy+b}{cy-a}$

$\Rightarrow f^{-1}(y) = \frac{-dy+b}{cy-a}$

$\therefore f^{-1}(x) = \frac{-dx+b}{cx-a}$

Exercise:

(i) Let the function f:R-{3} →R-{1} is defined by $f(x) = \frac{x-2}{x-3}$. Find a formula which defined f^{-1}.

(ii) Let the function f: **R** – {$\frac{1}{2}$} → R-{$\frac{1}{2}$} is defined by f(x) =$\frac{x-3}{2x-1}$.

Find the domain and range of the finction. Evaluate $f^{-1}(2)$

(iii) If $f(x) = \sqrt{x-2}$ then find $f^{-1}(x)$. Also sketch the graph of $f(x)$ and $f^{-1}(x)$.

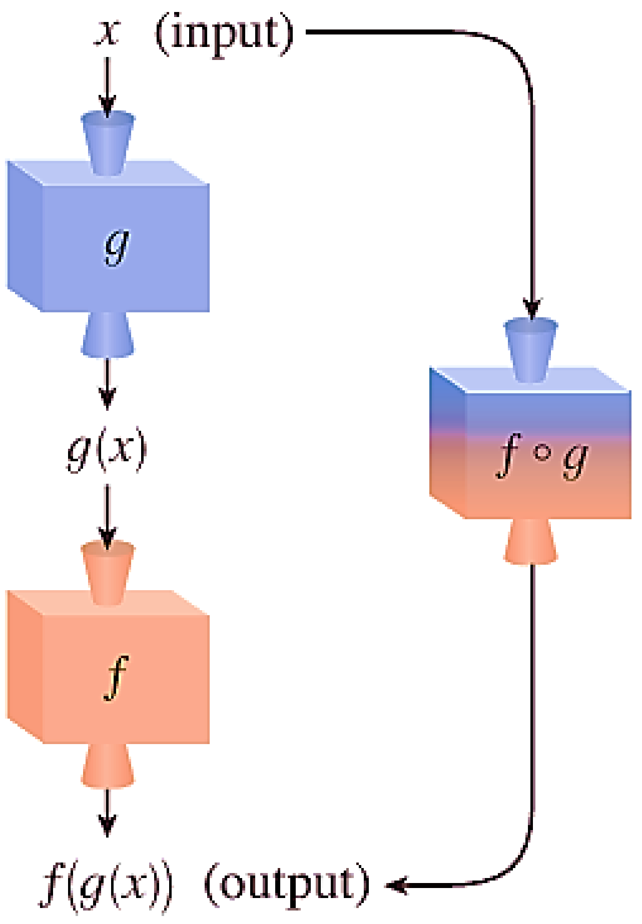
(iv) The function f: **R** → R is defined by $f(x) = (x+1)^2$. Find the domain and range of the function. Evaluate $f^{-1}(4)$

Composite Function:

Let $f: A \to B$ & $g: B \to C$ where $R_f = B = D_g$ then the composite function of f and g is denoted by $f \circ g$ (also called the composition of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined



The $f o g$ function is composed of the g function (first) and then the f function.

Question: .

If $f(x) = x^2$ and $g(x) = x - 3$, find the composite function $f \circ g$ and $g o f$.

Solution:

We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)2$$

And $(g o f)(x) = g(f(x)) = g(x^2) = x^2 - 3$

Question: .

Let $f(x)=\sqrt{x}$, $g(x) = \sqrt{2-x}$ Find

(i) $(fof)(x)$;

(ii) $(gog)(x)$

(iii) Domain & range of $(gog)(x)$

(iv) $(gof)(4)$;

Solution:

Given that

$f(X) = \sqrt{x}$, and $g(x) = \sqrt{2-x}$;

(i) $(fof)(x) = f(f(X))$

$= f(\sqrt{x})$

$= \sqrt{\sqrt{x}}$

$= x^{\frac{1}{4}}$

(ii) $(gog)(x) = g(g(x))$

$= g(\sqrt{2-x})$

$= \sqrt{2-\sqrt{2-x}}$

(iii) **Domain & range of $(gog)(x)$:**

Here, $(gog)(x)=\sqrt{2-\sqrt{2-x}}$

so, $2-\sqrt{2-x}) \geq 0$

$\Rightarrow \sqrt{2-x} \leq 2$

$\Rightarrow 2-x \leq 4$

$\Rightarrow x \geq -2$

and

also $2-x \geq 0$

$\Rightarrow x \leq 2$

$\therefore Domain= [-2,2]$

Now ,

if $x=-2$ then, $(gog)(x)=\sqrt{2-\sqrt{2+2}} = 0$

if $x = 2$ then, $(gog)(x)=\sqrt{2-\sqrt{2-2}} = \sqrt(2)$

if $x = 0$ then, $(gog)(x)=\sqrt{2-\sqrt{2-0}}=\sqrt{2-\sqrt{2}}$

$\therefore$Maximum value of $(gog)(x) = \sqrt{2}$

and Minimum value of $(gog)(x) = 0;$

$\therefore$*Range*=$[0, \sqrt{2}]$

(iv) $(gof)(4) = g(f(4))$

$= g(\sqrt{4})$

=g(2)

$= \sqrt{2 - \sqrt{2 - 2}}$

$= \sqrt{2 - 0}$

$= \sqrt{2};$

Exercise:

(i) If $f(x) = x^2 - 16$ & $g(x) = \sqrt{x}$. Then find $(gof)(x)$

Hence find the domain of $(gof)(x)$.

(ii) If $f(x) = 2x^3 + 3$ & $g(x) = \sqrt[3]{\frac{x-3}{2}}$. Then show that

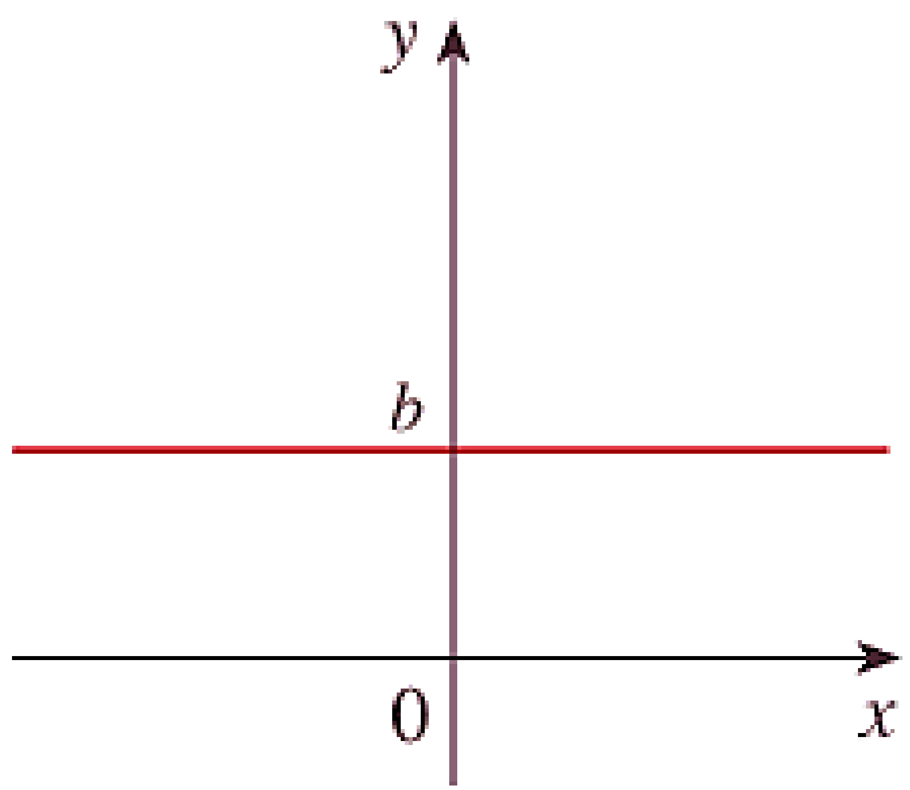
$(fog)(x) = (gof)(x)$

(iii) If $f(x) = \frac{1}{1+x}$. Then find that $f\,f\big(f(x)\big)$.

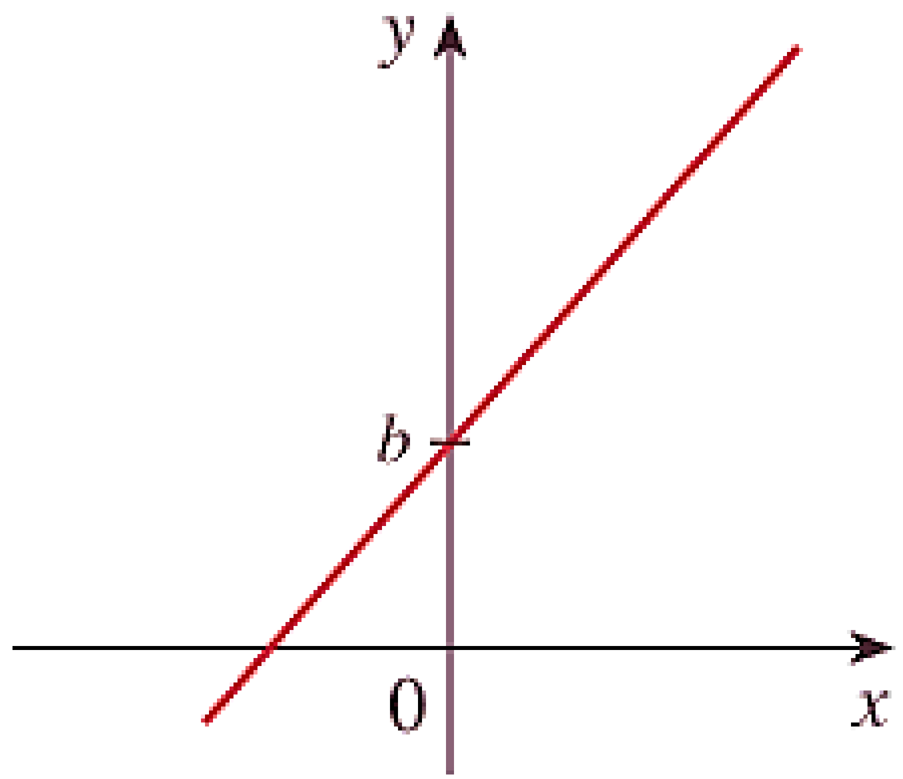
Families of Essential Functions and Their Graphs

Linear Functions

$$f(x) = mx + b$$



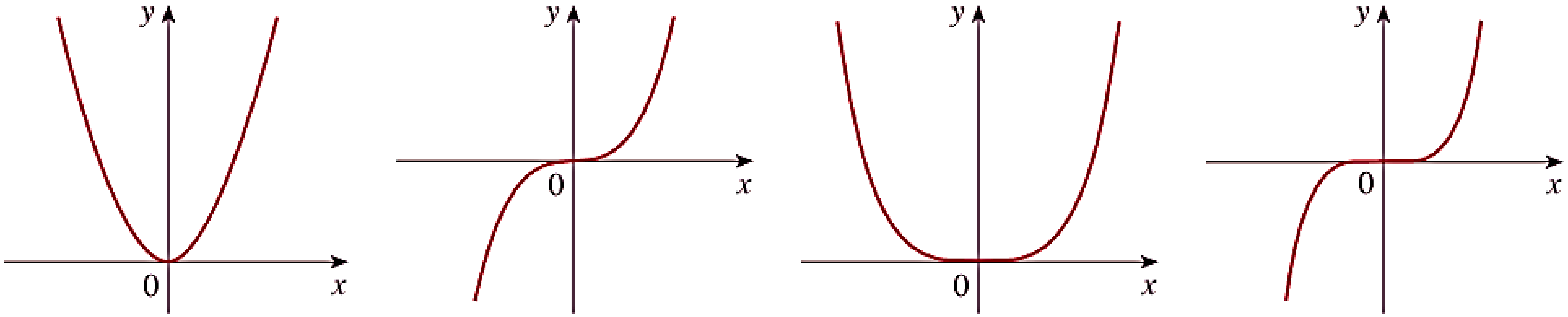
$f(x) = b$



$f(x) = mx + b$

Power Functions

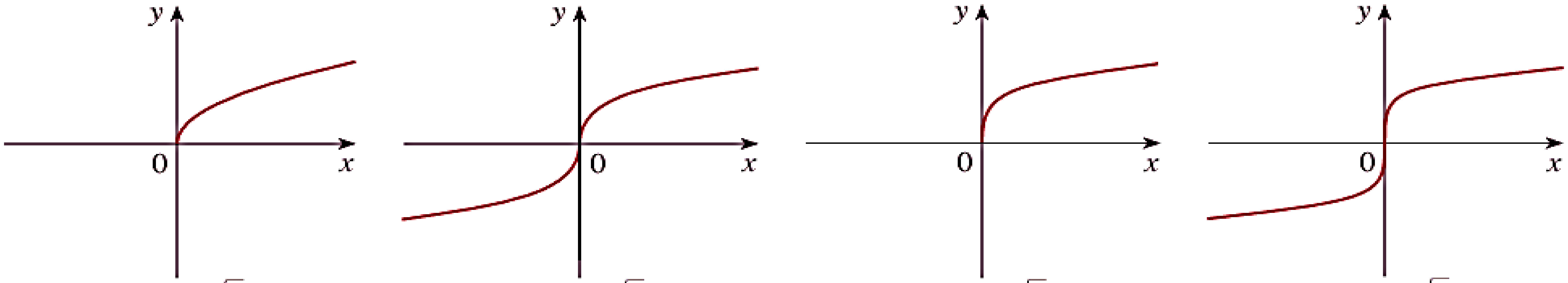
$$f(x) = x^n$$



$f(x) = x^2$ $f(x) = x^3$ $f(x) = x^4$ $f(x) = x^5$

Root Functions

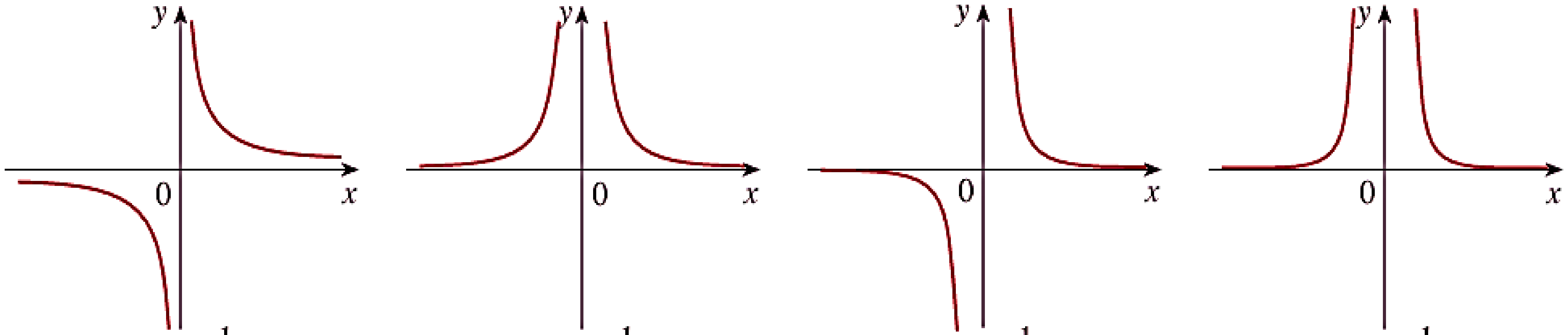
$$f(x) = \sqrt[n]{x}$$



$f(x) = \sqrt{x}$ $f(x) = \sqrt[3]{x}$ $f(x) = \sqrt[4]{x}$ $f(x) = \sqrt[5]{x}$

Reciprocal Functions

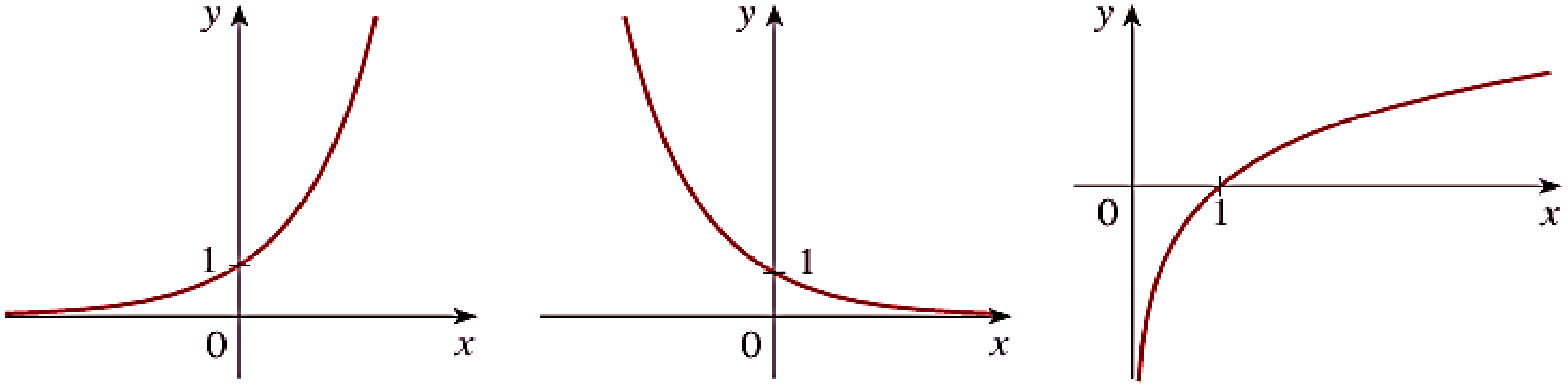
$$f(x) = \frac{1}{x^n}$$



$f(x) = \frac{1}{x}$ $f(x) = \frac{1}{x^2}$ $f(x) = \frac{1}{x^3}$ $f(x) = \frac{1}{x^4}$

Exponential and Logarithmic Functions

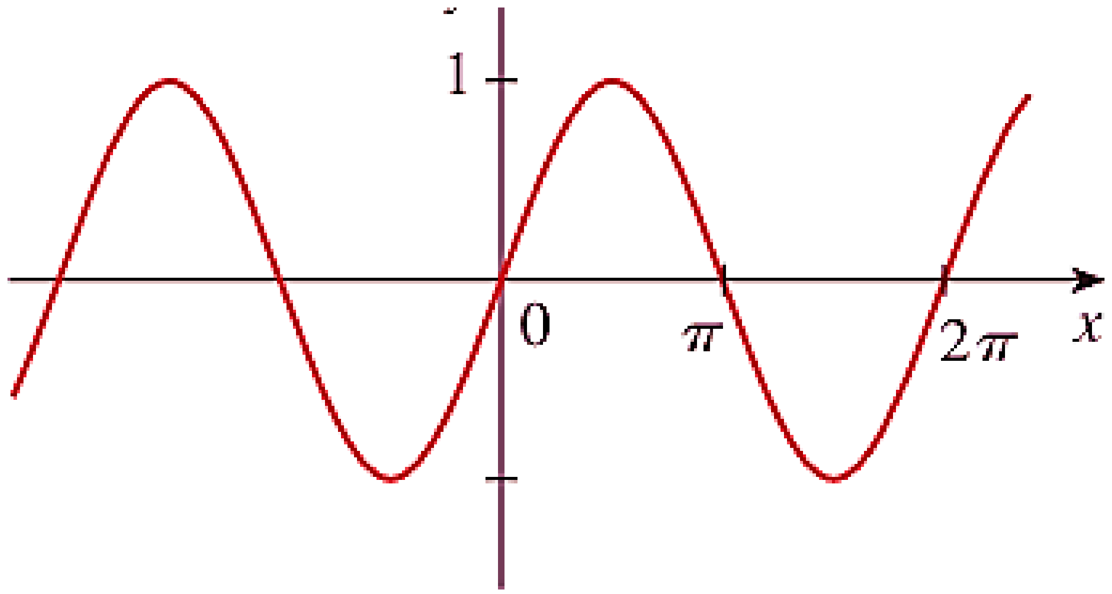
$f(x) = b^x$ and $f(x) \log_b x$



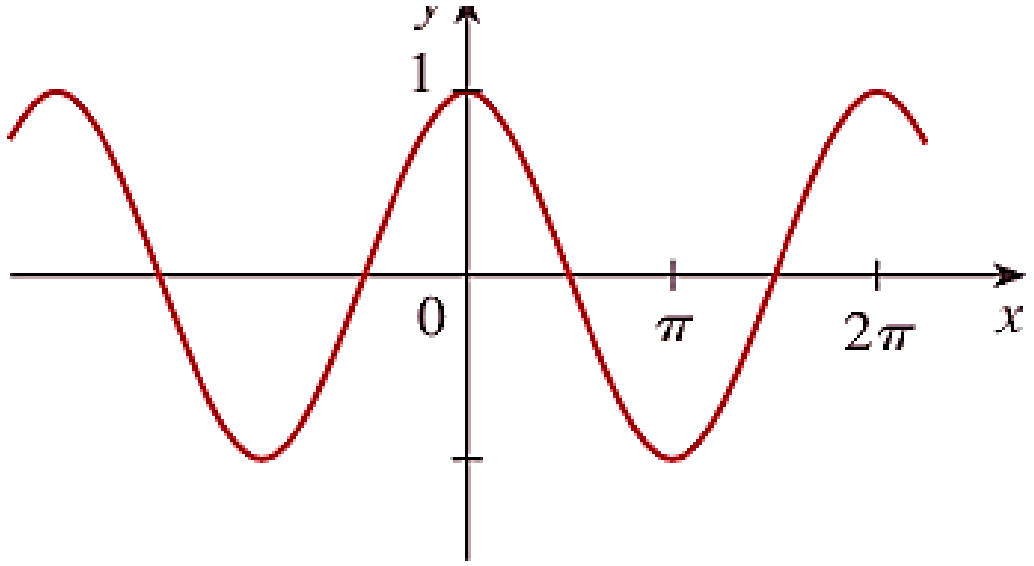
$f(x) = b^x \quad (b > 1)$ $f(x) = b^x \quad (b < 1)$ $f(x) = \log_b x \quad (b > 1)$

Trigonometric Functions

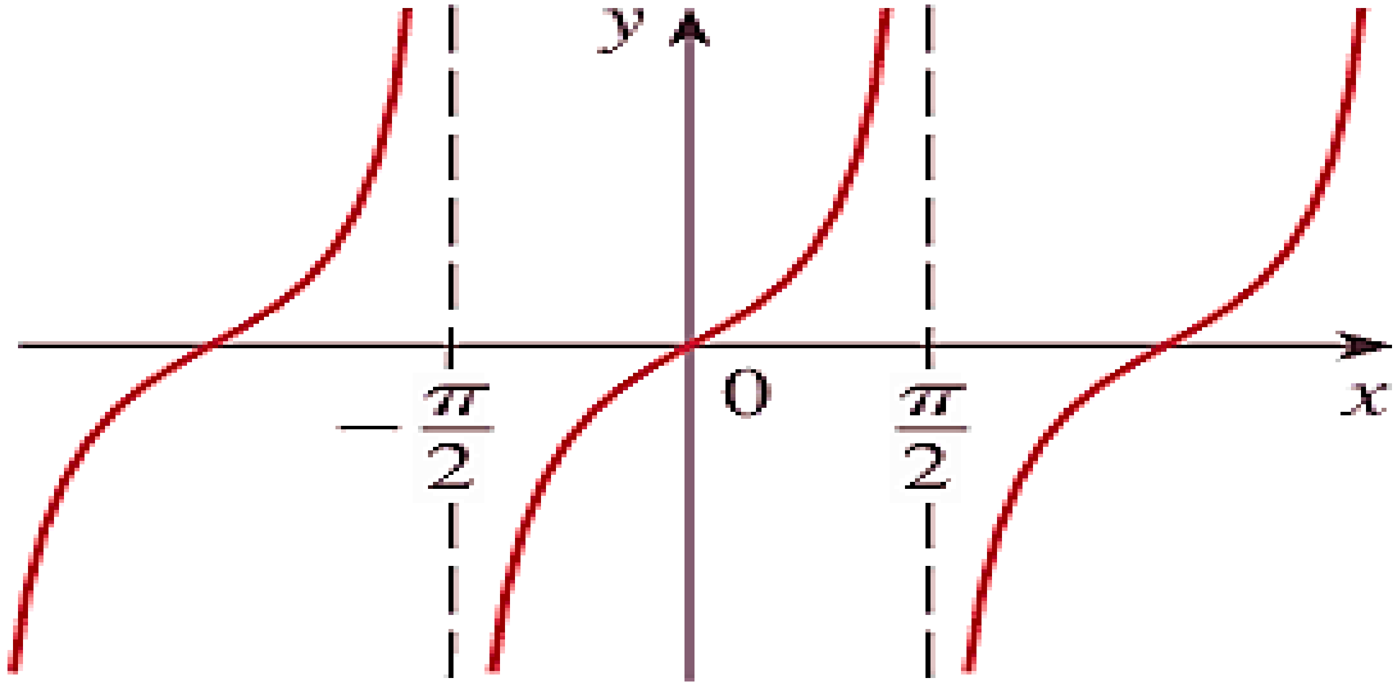
$$f(x) = sinx, cosx, tanx$$



$f(x) = \sin x$



$f(x) = \cos x$



$f(x) = \tan x$

Transformations of Functions

Vertical and Horizontal Shifts:

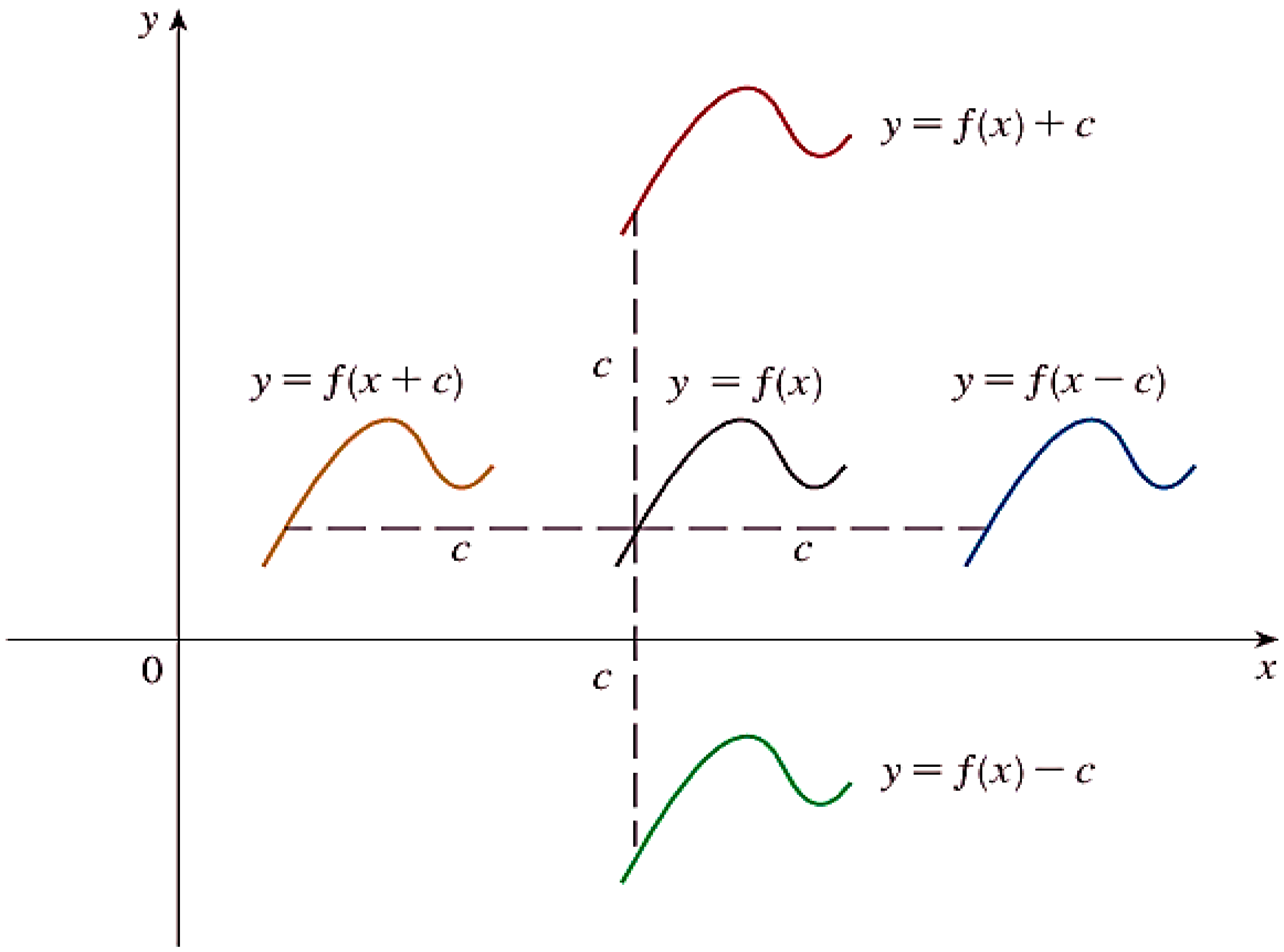
Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of y = f(x) a distance c units **upward**

$y = f(x) - c$, shift the graph of y = f(x) a distance c units **downward**

$y = f(x + c)$, shift the graph of y = f(x) a distance c units **to the left**

$y = f(x - c)$, shift the graph of y = f(x) a distance c units **to the right**



Translating the graph of f

Vertical and Horizontal Stretching and Reflecting:

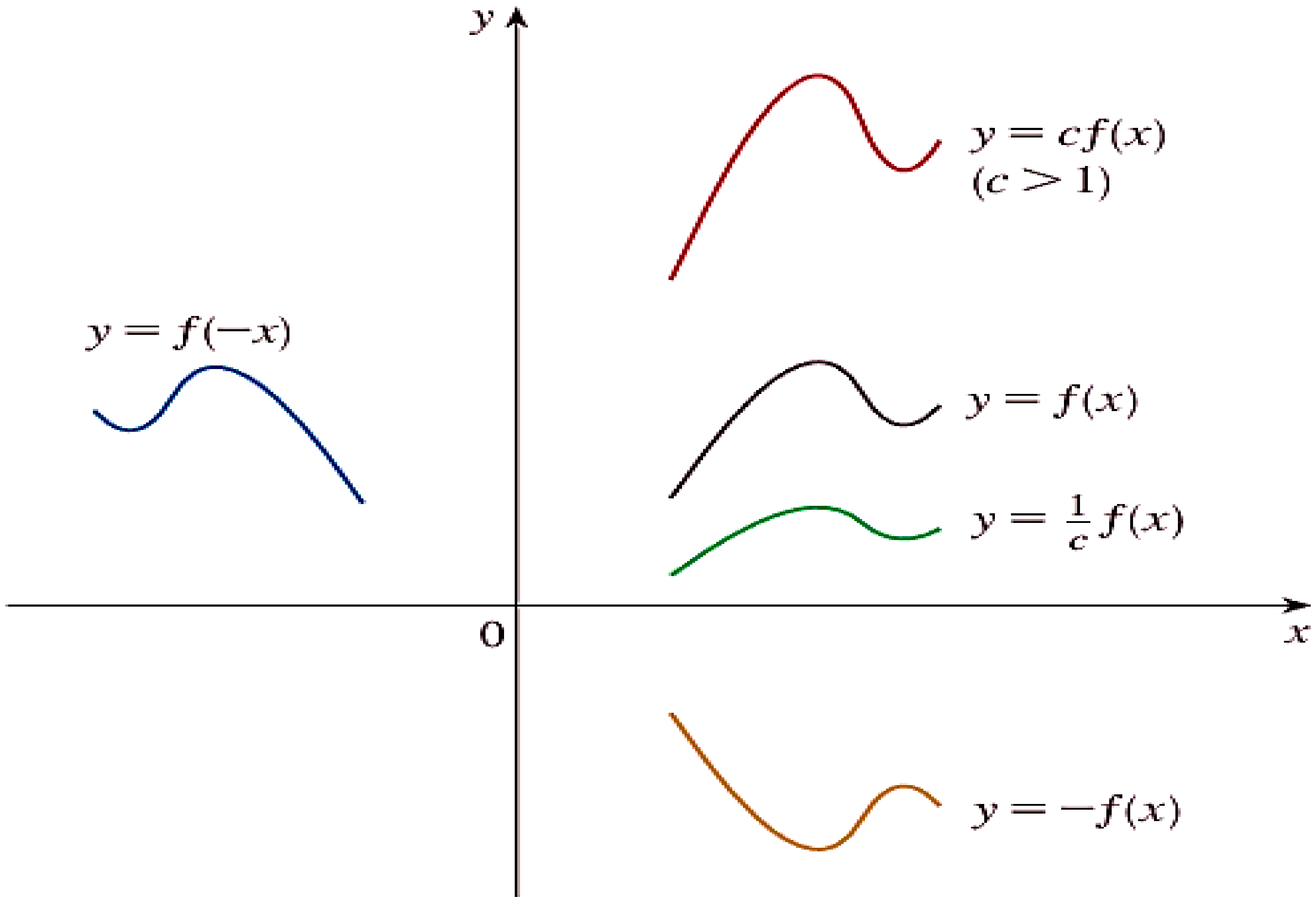
Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, **stretch** the graph of y = f(x) vertical by a factor of c

$y = (1/c)f(x)$, **shrink** the graph of y = f(x) vertically by a factor of c

$y = -f(x)$, **reflect** the graph of y = f(x) about th x-axis

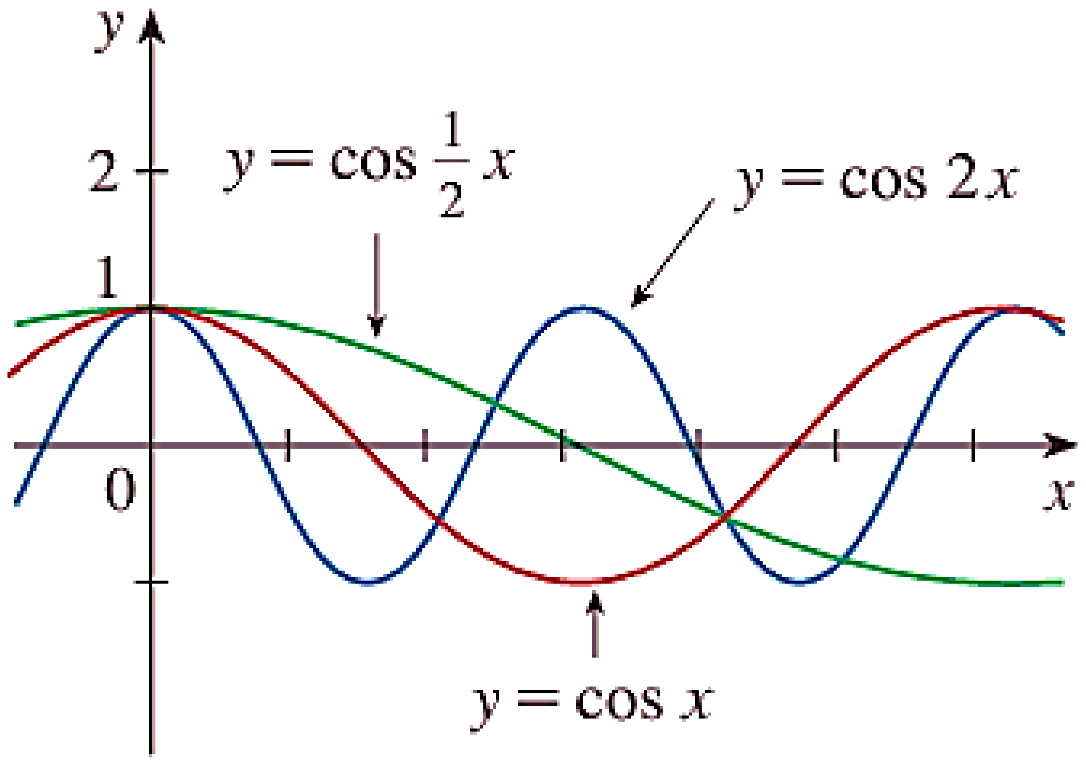
$y = f(-x)$, **reflect** the graph of y = f(x) about th y-axis



Stretching and reflecting the graph of f

$y = f(cx)$, shrink the graph of y = f(x) horizontally by a factor of c

$y = f(x/c)$, stretch the graph of y = f(x) horizontally by a factor of c



Stretching and reflecting the graph of f

WEAK 4

Page:17-25

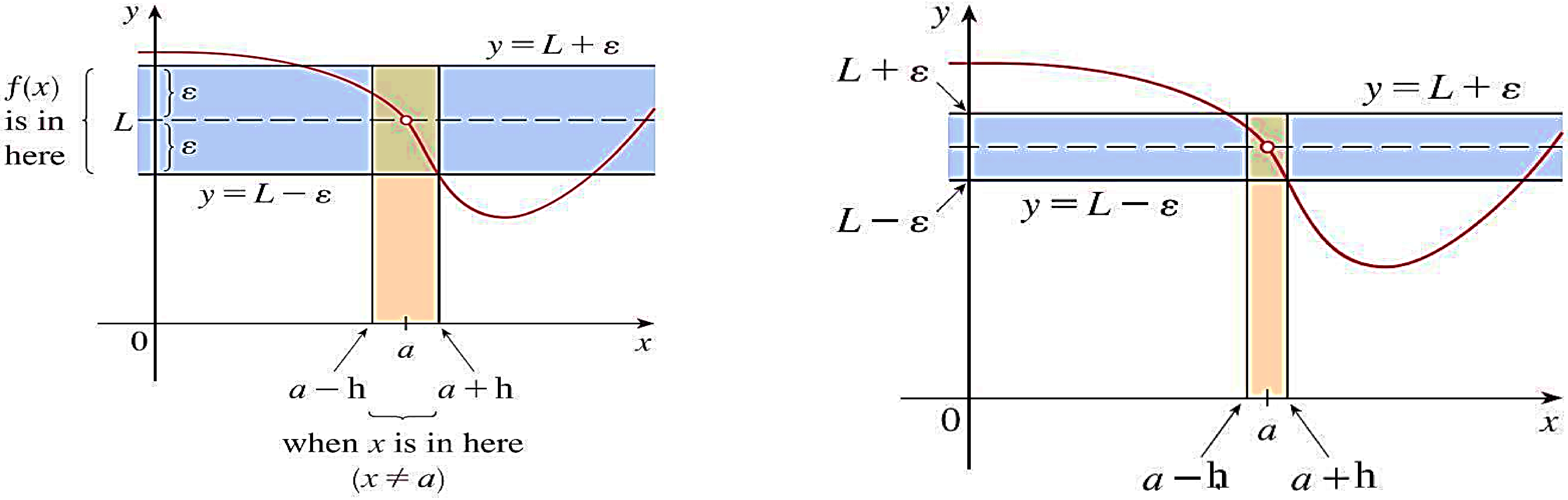
LIMIT OF A FUNCTION

LIMIT:

If the values of $f(x)$ can be made as close as we like to 'L' by taking values of x sufficient close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x) = L$$

Which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a".



Graphical representation of "limit of a function" at a point

Left Hand Limit:

If the values of $f(x)$ can be made as close as we like to "L" by talking values of x sufficiently close to "a" (but less than a)

Then we write

$$\text{L.H.L.} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = L$$

Right hand limit:

If the values of $f(x)$ can be made as close as we like to "L" by talking values of x sufficiently close to "a" (but greater than a)

Then we write

$$\text{R.H.L} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = h$$

Fundamental properties of limit:

$$(i)\ \lim_{x\to a}\{f(x) \pm g(x)\} = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$(ii)\ \lim_{x\to a}\{f(x) \times g(x)\} = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$$

$$(iii)\ \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\ ;\ g(a) \neq 0$$

$$(iv)\ \lim_{x\to a}\{cf(x)\} = c \lim_{x\to a} f(x)$$

$$(v)\ \lim_{x\to a}\{f(x)\}^n = [\lim_{x\to a} f(x)]^n$$

Some important formulae:

- $\lim_{x \to a} \frac{x^n - a^n}{x - a} = \text{n}a^{n-1}$
- $\lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim_{x \to 0} \frac{x}{\tan x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = 1$
- $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$
- $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$
- $\lim_{x \to \infty} (1 + \frac{1}{x})^x = \text{e}$
- $\lim_{x \to 0} \frac{a^x - 1}{x} = \ln(\text{a})$

- $1+2+3+\ldots\ldots\ldots\ldots+\text{n} = \frac{n(n+1)}{2}$
- $1^2+2^2+3^2+\ldots\ldots\ldots+\text{n} = \frac{n(n+1)(2n+1)}{6}$
- $1^3+2^3+3^3+\ldots\ldots\ldots+\text{n} = \left\{\frac{n(n+1)}{2}\right\}^2$
- $1 - \cos 2\theta = 2sin^2\theta$

Question: Find the value:

(i) $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+x-6}$

Solution:

$\lim_{x\to 2}\frac{x^2-4x+4}{x^2+x-6}$

$=\lim_{x\to 2}\frac{x^2-2.x.2+2^2}{x^2+3x-2x-6}$

$=\lim_{x\to 2}\frac{(x-2)^2}{x(x+3)-2(x+3)}$

$=\lim_{x\to 2}\frac{(x-2)^2}{(x+3)(x-2)}$

$=\lim_{x\to 2}\frac{(x-2)}{x+3}$

$=\frac{2-2}{2+3}$

$=0$

Question: Find the value:

(ii) $\lim_{x\to \infty}\frac{3x^3+20x^2+2}{6-20x^2-4x^3}$

Solution:

$\lim_{x\to \infty}\frac{3x^3+20x^2+2}{6-20x^2-4x^3}$

$=\lim_{x\to \infty}\frac{x^3(3+\frac{20}{x}+\frac{2}{x^3})}{x^3(\frac{6}{x^3}-\frac{20}{x}-4)}$

$=\lim_{x\to \infty}\frac{3+\frac{20}{x}+\frac{2}{x^3}}{\frac{6}{x^3}-\frac{20}{x}-4}$

$=\frac{3+\frac{20}{\infty}+\frac{2}{\infty}}{\frac{6}{\infty}-\frac{20}{\infty}-4}$

$=\frac{3+0+0}{0-0-4}$

$=-\frac{3}{4}$

(iii) $\lim_{x \to -\infty} \frac{\sqrt{4x^2-2}}{3x+1}$

Solution:

$$\lim_{x \to -\infty} \frac{\sqrt{4x^2-2}}{3x+1}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2(4-\frac{2}{x^2})}}{x(3+\frac{1}{x})}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{4-\frac{2}{x^2}}}{x(3+\frac{1}{x})}$$

$$= \lim_{x \to -\infty} \frac{-x\sqrt{4-\frac{2}{x^2}}}{x\left(3+\frac{1}{x}\right)}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{4-\frac{2}{x^2}}}{3+\frac{1}{x}}$$ [since $\sqrt{x^2} = |x| = \{-x; x < 0\}$]

$$= \lim_{x \to -\infty} \frac{-\sqrt{4-\frac{2}{x^2}}}{3+\frac{1}{x}}$$

$$= \frac{-\sqrt{4-\frac{2}{\infty}}}{3+\frac{1}{\infty}}$$

$$= \frac{-\sqrt{4}}{4}$$

$$= -\frac{2}{3}$$

(iv) $\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

Solution:

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

$$=\lim_{n\to\infty} \frac{n(n+1)(2n+)}{6n^3}$$

$$=\lim_{n\to\infty} \frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6n^3}$$

$$=\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)(2+\frac{1}{n})}{6}$$

$$=\frac{(1+\frac{1}{\infty})(2+\frac{1}{\infty})}{6}$$

$$=\frac{2}{6}$$

$$=\frac{1}{3}$$

(v) $\text{im}_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

Solution:

$$\text{im}_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

$$=\lim_{n\to a} \frac{\frac{x^{\frac{3}{7}}-a^{\frac{3}{7}}}{x-a}}{\frac{x^{\frac{2}{5}}-a^{\frac{2}{5}}}{x-a}}$$

$$=\frac{\lim_{n\to\infty} \frac{x^{\frac{3}{7}}-a^{\frac{3}{7}}}{x-a}}{\lim_{n\to a} \frac{x^{\frac{2}{5}}-a^{\frac{2}{5}}}{x-a}}$$

$$=\frac{\frac{3}{7}a^{\frac{3}{7}-1}}{\frac{2}{5}.a^{\frac{2}{5}-1}}$$

$$=\frac{3}{7}\times\frac{5}{2}a^{\left(\frac{3}{7}-1\right)-\left(\frac{2}{5}-1\right)}=\frac{15}{14}a^{\frac{3}{7}-\frac{2}{5}}$$

$$=\frac{15}{14}a^{\frac{3}{7}-\frac{2}{5}}=\frac{15}{14}a^{\frac{1}{35}}$$

(v) $\lim_{n\to\infty} \frac{5^{n+1}+7^{n+1}}{5^n-7^n}$

Solution:

$$\lim_{n\to\infty} \frac{5^{n+1}+7^{n+1}}{5^n-7^n}$$

$$=\lim_{n\to\infty} \frac{5^n.5+7^n.7}{5^n-7^n}$$

$$=\lim_{n\to\infty} \frac{7^n\{\left(\frac{5}{7}\right)^n.5+7\}}{7^n\{\left(\frac{5}{7}\right)^n-1\}}$$

$$=\lim_{n\to\infty} \frac{(\frac{5}{7})^n.5+7}{(\frac{5}{7})^n-1}$$

$$=\frac{(\frac{5}{7})^\infty.5+7}{(\frac{5}{7})^\infty-1}$$

$$=\frac{0+7}{0-1}, \quad \left[\text{since } 0<\frac{5}{7}<1 \ \therefore \left(\frac{5}{7}\right)^\infty=0\right]$$

$$=-7$$

(vi) $\lim_{x\to 0} \frac{1-\cos 7x}{3x^2}$

Solution:

$$\lim_{x\to 0} \frac{1-\cos 7x}{3x^2}$$

$$=\lim_{x\to 0} \frac{2\sin^2\frac{7x}{2}}{3x^2}$$

$$=\frac{2}{3}\lim_{x\to 0}\left(\frac{\sin\frac{7x}{2}}{\frac{7x}{2}}\right)^2\times\frac{49}{4}$$

$$=\frac{2}{3}\times 1\times\frac{49}{4}$$

$$=\frac{49}{6}$$

(vii) $\lim_{x\to 0}\frac{cos7x-cos9x}{cos3x-cos5x}$

Solution:

$$\lim_{x\to 0}\frac{cos7x-cos9x}{cos3x-cos5x}$$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{7x+9x}{2}\right).\sin\left(\frac{9x-7x}{2}\right)}{2\sin\left(\frac{3x+5x}{2}\right).\sin\left(\frac{5x-3x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{\sin 8x.sinx}{sin4x.sinx}$$

$$=\lim_{x\to 0}\frac{2sin4x.cos4x}{sin4x}\qquad [\because sin2A = 2sinA.cosA]$$

$$=\lim_{x\to 0}(2\cos 4x)$$

$$= 2\cos 0$$

$$= 2\times 1$$

$$= 2$$

(viii) $\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}$

Solution:

$$\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{(3-\sqrt{x^2+5})(3+\sqrt{x^2+5})}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{3^2-(\sqrt{x^2+5})}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(\sqrt{x^2+5})}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{(4-x^2)}$$

$$=\lim_{x\to 2}(3+\sqrt{x^2+5})$$

$$= (3+\sqrt{x^2+5})$$

$$= 3+\sqrt{9} = 6$$

(ix) $\lim_{x\to 0}\frac{tanx-\sin x}{sin^3x}$

Solution:

$$\lim_{x\to 0}\frac{tanx-\sin x}{sin^3x}$$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{sin^3x}$$

$$=\lim_{x\to 0}\frac{\sin(\frac{1}{cosx}-1)}{sin^3x}$$

$$=\lim_{x\to 0}\frac{\frac{1-cosx}{\cos x}}{sin^2x}$$

$$=\lim_{x\to 0}\frac{1-cosx}{cosx(1-cos^2x)}$$

$$=\lim_{x\to 0}\frac{(1-cosx)}{\cos x.(1-cosx)(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{1}{\cos x(1+\cos x)}$$

$$=\frac{1}{\cos 0(1+\cos 0)}$$

$$=\frac{1}{1(1+1)}=\frac{1}{2}$$

(x) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x}$

Solution:

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x}$$

$$=\lim_{x\to 0}\frac{(e^x-1)+(e^x-1)}{x}$$

$$=\lim_{x\to 0}[\frac{e^x-1}{x}+\frac{e^x-1}{x}]$$

$$=\lim_{x\to 0}\frac{e^x-1}{x}+\lim_{x\to 0}\frac{e^x-1}{x}$$

$$=1+1$$

$$=2$$

EXERCISE:

(i) $\lim\limits_{x \to 1} \frac{x^3+x^2-5x+3}{x^3-3x+2}$

(ii) $\lim\limits_{x \to \infty} \left(\sqrt{x^2+x} - x\right)$

(iii) $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \frac{\cos 3x}{x^2}\right)$

(iv) $\lim\limits_{x \to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

(v) $\lim\limits_{x \to 0} \frac{1-2\cos x+\cos 2x}{x^2}$

(vi) $\lim\limits_{x \to \frac{\pi}{2}} \frac{\sec^3 x-\tan^3 x}{\tan x}$

(vii) $\lim\limits_{x \to 0} \frac{\cos 3x-\cos 5x}{x^2}$

WEAK 5

Page:17-25

Existence of limit of a function f(x) at $x = a$.

The limit of a function f(x) at x=a that is $\lim_{x \to a} f(x)$ exists if

(i) L.H.L = $\lim_{x \to a^-} f(x) = L$ (exists)

(ii) R.H.L = $\lim_{x \to a^+} f(x) = L$ (exists)

(iii) L.H.L = R.H.L

Left Hand Limit:

If the values of $f(x)$ can be made as close as we like to "L" by talking values of x sufficiently close to "a" (but less than a)

Then we write

$$\text{L.H.L.} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h) = L$$

Right hand limit:

If the values of $f(x)$ can be made as close as we like to "L" by talking values of x sufficiently close to "a" (but greater than a)

Then we write

$$\text{R.H.L} = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h) = L$$

Question:

Let, f(x) = $\begin{cases} \frac{1}{x+2} & ; x < -2 \\ x^2 - 5 & ; -2 < x \le 3 \\ \sqrt{x+13} & ; x > 3 \end{cases}$

Find (i) $\lim_{x \to 2} f(x)$ (ii) $\lim_{x \to 0} f(x)$ (iii) $\lim_{x \to 3} f(x)$

Or does $\lim_{x \to 3} f(x)$ exist? If exist then find it.

Solution:

Given that, f(x) = $\begin{cases} \frac{1}{x+2} ; & x < -2 \\ x - 5; & -2 < x \le 3 \\ \sqrt{x+13}; & x > 3 \end{cases}$

(i) If $x \to -2^-$ then $f(x) = \frac{1}{x+2}$

$\therefore \lim_{x \to -2^-} f(x) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} \frac{1}{(-2-h)+2} = \lim_{x \to 0} \frac{1}{-h} = (-1) \times \frac{1}{0} = -\infty$

If $x \to -2^-$ then $f(x) = x^2 - 5$

$\therefore \lim_{x \to -2^+} f(x) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} \{(-2+h)^2 - 5\} = (-2+0)^2 - 5 = -1$

Since, $\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x)$

So $\lim_{x\to -2} f(x)$ *does not exist*.

(ii) If $x \to 0^+$ or $x \to 0^-$ then $f(x) = x^2 - 5$

$\therefore \lim_{x\to 0} f(x) = \lim_{h\to 0} f(x^2 - 5) = 0 - 5 = -5$

(iii) IF $x \to 3^-$ then $f(x) = x^2 - 5$

$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2 - 5) = 3^2 - 5 = 4$

If $x \to 3^+$ then $f(x) = \sqrt{x + 13}$

$\therefore \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left(\sqrt{x + 13}\right) = \lim_{x\to 3^+} \left(\sqrt{3 + 13}\right) = 4$

Since, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 4$

$\therefore \lim_{x\to 3} f(x) = 4$

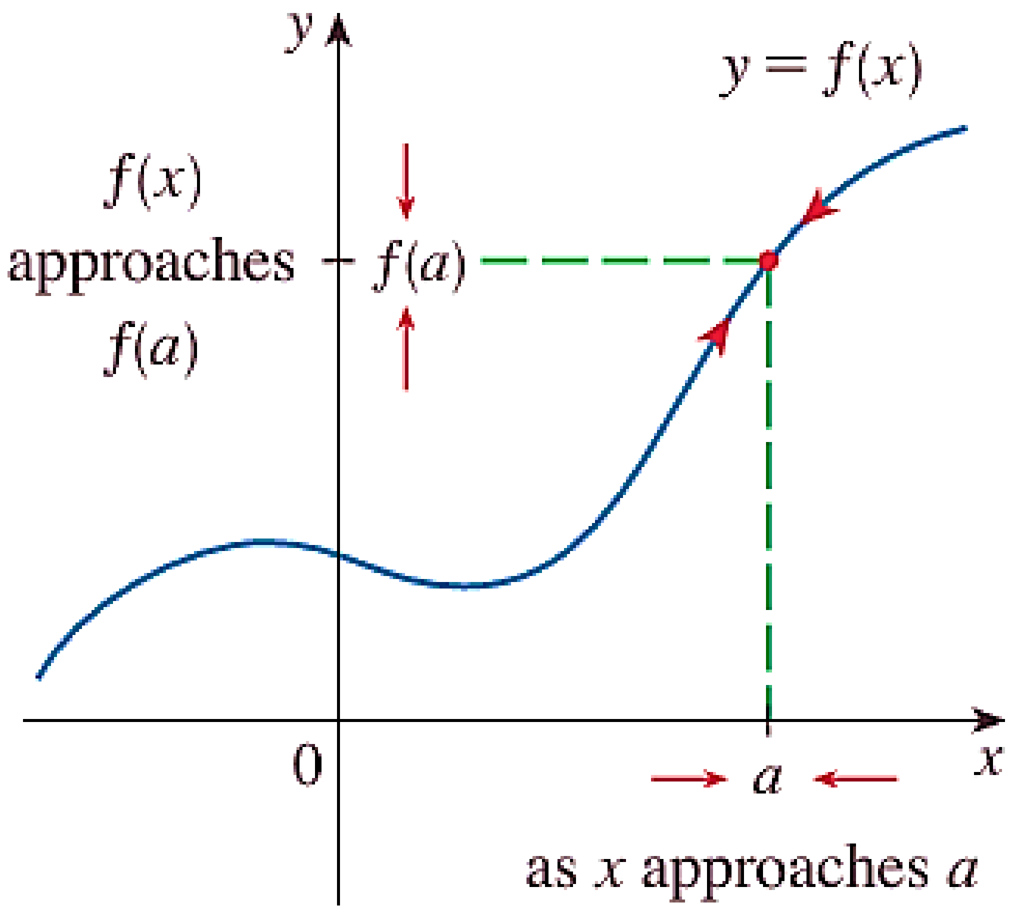
CONTINUITY:

A function f is said to be continuous function at $x = a$ provided the following conditions are satisfied:

(i) f(a) is defined

(ii) $\lim_{x \to a} f(x)$ exists

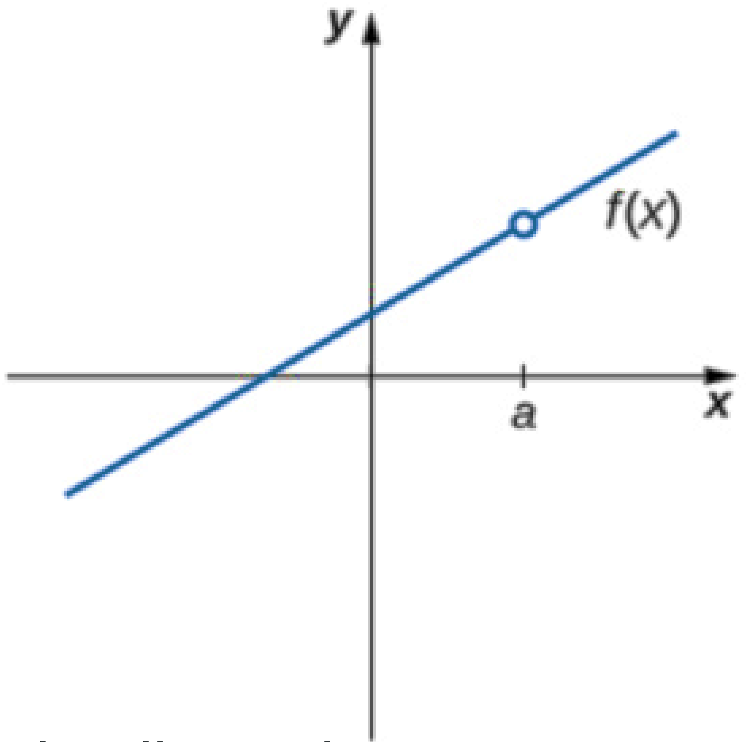
(iii) $\lim_{x \to a} f(x) = f(a)$ i.e. $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$



f is continuous at $x = a$

DISCONTINUITY:

If $f(x)$ is not continuous at $x = a$ then it is called discontinuous



f is discontinuous at $x = a$

Question:

The real function f is given by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & ; x \neq 2 \\ 3 & ; x = 2 \end{cases}$$

Show that the function f is discontinuous at $x = 2$. Define the function f is such a way that it is continuous at $x = 2$

Solution:

Given that $f(x) = \begin{cases} \frac{x^2-4}{x-2} & ; x \neq 2 \\ 3 & ; x = 2 \end{cases}$

When $x = 2$ then $f(x) = 3$

$$\therefore f(2) = 3$$

$$\lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h)$$

$$= \lim_{h \to 0} \frac{(2-h)^2 - 4}{(2-h) - 2}$$

$$= \lim_{h \to 0} \frac{(2-h+2)(2-h-2)}{(2-h-2)}$$

$$= \lim_{h \to 0} (4 - h)$$

$$= 4 - 0 = 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 4}{(2+h) - 2}$$

$$= \lim_{h \to 0} \frac{(2+h+2)(2+h-2)}{(2+h-2)}$$

$$= \lim_{h \to 0} (4 + h)$$

$$= 4 + 0 = 4$$

$$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \neq f(2)$$

So that function $f(x)$ is discontinuous at $x = 2$.

2nd part:

For the continuity f(x) is defined at $x = 2$ in the following way.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & ; x \neq 2 \\ 4 & ; x = 2 \end{cases}$$

Question:

At $x = 0$ and $x = 1$ discontinuous the continuity of the function

f : $\mathbb{R} \to R$ where,

$$f(x) = \begin{cases} x^2 + 1 & ; x < \\ x & ; 0 \le x \le 1 \\ \frac{1}{x} & ; x > 1 \end{cases}$$

Solution:

Given that $f(x) = \begin{cases} x^2 + 1 & ; x < \\ x & ; 0 \le x \le 1 \\ \frac{1}{x} & ; x > 1 \end{cases}$

At $x = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 1) = 0 + 1 = 1$$

$$\therefore \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So, $f(x)$ is discontinuous at $x = 0$

At $x = 1$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{1}{x} = \frac{1}{1} = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1 = 1$$

When $x = 1$ then $f(x) = x$. $\therefore f(1) = 1 \therefore$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} (x) = f(10$$

So, $f(x)$ is continuous at $x = 1$

Question:

$$\text{Let,} f(x) = \begin{cases} 1+2x & ; -\frac{1}{2} \le x < 0 \\ 1-2x & ; 0 \le x \le \frac{1}{2} \\ -1+2x & ; -x \ge \frac{1}{2} \end{cases}$$

Find $\lim_{x\to 0} f(x)$ & $\lim_{x\to \frac{1}{2}} f(x)$ and the test the continuity of the function f(x) at $x = 0$ and $x = \frac{1}{2}$

Solution:

$$\text{Given that, } f(x) = \begin{cases} 1+2x & ; -\frac{1}{2} \le x < 0 \\ 1-2x & ; 0 \le x \le \frac{1}{2} \\ -1+2x & ; -x \ge \frac{1}{2} \end{cases}$$

Here, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(1-2x) = 1-0 = 1$ And $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} f(1+2x) = 1-(2\times 0) = 1$

$$\therefore \lim_{x\to 0} f(x) = 1$$

Again, if $x = 0$ then $f(x) = 1-2x$

$$\therefore f(0) = 1-2\times 0 = 1$$

$$\therefore \lim_{x\to 0} f(x) = f(0)$$

So, $f(x)$ is continuous at $x = 0$

Again, $\lim_{x\to(\frac{1}{2})^+} f(x) = \lim_{x\to(\frac{1}{2})^+} (-1+2x)$

$= -1 + \frac{1}{2} \times 2$

$= 0$

And $\lim_{x\to(\frac{1}{2})^-} f(x) = \lim_{x\to(\frac{1}{2})^-} (1+2x)$

$= 1 - \frac{1}{2} \times 2$

$= 0$

$\therefore \lim_{x\to\frac{1}{2}} f(x) = 0$

Again, if $x = 0$ then $f(x) = -1 + 2x$ $\therefore f(\frac{1}{2}) = -1 + \frac{1}{2} \times 2 = 0$

$\therefore \lim_{x\to\frac{1}{2}} f(x) = f\left(\frac{1}{2}\right)$

So, $f(x)$ is continuous at $x = \frac{1}{2}$

Exercises:

(i) If $f(x) = \frac{\sqrt{2}cosx-1}{cotx-1}$; $x \neq \frac{\Pi}{4}$ Find the value of $f(\frac{\Pi}{4})$ so that f(x) becomes continuous at $x = \frac{\Pi}{4}$. Ans: $\frac{1}{2}$

(ii) Show that the function f given by f(x)=$\begin{cases} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & ; if\ x \neq 0 \\ 0 & ; if\ x = 0 \end{cases}$ is discontinuous at $x = 0$

(iii) A function $f(x)$ is defined as follows f(x)=$\begin{cases} -x & ; x \leq 0 \\ x & ; 0 < x < 1 \\ 1-x & ; x \geq 1 \end{cases}$ Discuss the continuity at $x = 1$

(iv) Let, f(x)=$\begin{cases} 1 & ; -\infty < x < 0 \\ 1 + sinx & ; 0 \leq x \leq \frac{\Pi}{2} \\ 2 + (x - \frac{\Pi}{2})^2 & ; \frac{\Pi}{2} \leq x < \infty \end{cases}$ test the continuity at x=0 & $\frac{\Pi}{2}$

(v) Let, $f(x)$=$\begin{cases} \frac{1}{x+2} & ; x < -2 \\ x^2 - 5 & ; -2 < x \leq 3 \\ \sqrt{x+13} & ; x > 3 \end{cases}$ Show that f(x) is continuous at $x = 3$. But discontinuous at $x = -2$

WEAK 6

Page:17-25

DIFFERENTIABILITY

Differentiability of a function:

The derivative of $y = f(x)$ with respect to x (for any particular value of x) is denoted by $f'(x)$ or $\frac{dy}{dx}$ and defined as,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+h) - f(x)}{h}$$

Existence of Derivative:

A function $y = f(x)$ is called differentiable at $x = a$ if the left hand derivative and right hand derivative both are Equal at this point that is,

$L.H.D = \lim_{h \to 0} \frac{f(a-h)-f(a)}{-h}$ and $R.H.D = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

are both exist and equal.

Question:

A function $f(x)$ is defined as follows $f(x) = \begin{cases} x^2+1 & when x \leq 0 \\ x & when 0 < x < 1 \\ \frac{1}{x} & when x \geq 1 \end{cases}$

Discuss the differentiability at $x = 0$ and $x = 1$

Solution:

Given that, $f(x) = \begin{cases} x^2+1 & when x \leq 0 \\ x & when 0 < x < 1 \\ \frac{1}{x} & when x \geq 1 \end{cases}$

For $x = 0$

$$L.H.D = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\{(-h)^2+1\}-(0^2+1)}{-h}$$

$$= \lim_{h\to 0} \frac{h^2+1-1}{-h}$$

$$= \lim_{h\to 0} \frac{h^2}{-h} = \lim_{h\to 0}(-h) = 0$$

$$R.H.D = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h-\{(0)^2+1\}}{h}$$

$$= \lim_{h\to 0} \frac{h-1}{h}$$

$$= \lim_{h\to 0} \left(1-\frac{1}{h}\right) = -\propto$$

$\because L.H.D \neq R.H.D.$

So, the function is not differentiable at $x = 0$

For $x = 1$

$$L.H.D = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{1-h-1}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h}$$

$$= \lim_{h \to 0} (1)$$

$$= 1$$

$$R.H.D = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{1-1-h}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{-1}{1+h}$$

$$= \frac{-1}{1+0}$$

$$= -1$$

Since , $L.H.D \neq R.H.D$

So the function is not differentiable at $x = 1$

DIFFERENTIATION

- The derivative is a mathematical operator, which measures the rate of change of a quantity relative to another quantity. The process of finding a derivative is called differentiation.
- There are many phenomena related changing quantities such as speed of a particle, inflation of currency, intensity of an earthquake and voltage of an electrical signal etc. in the world. In this chapter we will discuss about various techniques of derivative

Differential Coefficient → $\frac{dy}{dx}$ ← **Derivative**

Measures rate of change of y in terms of x.

Derivatives of Elementary Functions:

1. $\frac{d}{dx}(c) = 0$, Where c is a constant.

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(x^n) = nx^{n-1}$.

4. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$.

5. $\frac{d}{dx}(e^x) = e^x$.

6. $\frac{d}{dx}(a^x) = a^x lna$.

7. $\frac{d}{dx}(lnx) = \frac{1}{x}$.

8. $\frac{d}{dx}(sinx) = cosx$.

9. $\frac{d}{dx}(cosx) = -sinx$.

10. $\frac{d}{dx}(tanx) = sec^2x$.

11. $\frac{d}{dx}(\sec x) = \sec x \tan x$

12. $\frac{d}{dx}(\cot x) = -cosec^2x$

13. $\frac{d}{dx}(cosec\ x) = -cosec\ x\ cot\ x$

14. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

15. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

16. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

17. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

18. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

19. $\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

20. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

21. $\frac{d}{dx}(u^v) = u^v\frac{d}{dx}(v\ln u)$

22. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Where u and v are functions of x.

Sum and Difference Rules:

If f and g are differentiable at x, then so are $f + g$ and $f - g$ and

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

Example:

Find the differential coefficient $\frac{dy}{dx}$ of the following functions,

(i) $y = x^{\frac{4}{5}}$ (ii) $y = 3x^8 - 2x^5 - 5x + 8$

(iii) $y = \frac{\sqrt{x} - 2x}{\sqrt{x}}$

Solution:

$$(i)\ \frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{4}{5}}\right)$$
$$= \frac{4}{5}x^{\frac{4}{5}-1}$$
$$= \frac{4}{5}x^{\frac{-1}{5}}$$

$$(ii)\ \frac{dy}{dx} = \frac{d}{dx}(3x^8 - 2x^5 - 5x + 8)$$
$$= 24x^3 - 10x^4 - 5$$

$$(iii)\ \frac{dy}{dx}=\frac{d}{dx}\left[\frac{\sqrt{x}-2x}{\sqrt{x}}\right]=\frac{d}{dx}\left[\frac{\sqrt{x}}{\sqrt{x}}-\frac{2x}{\sqrt{x}}\right]=\frac{d}{dx}\left[1-\frac{2x}{x^{\frac{1}{2}}}\right]$$

$$=\frac{d}{dx}\left[1-2x.x^{\frac{-1}{2}}\right]=\frac{d}{dx}\left[1-2x^{1-\frac{1}{2}}\right]=\frac{d}{dx}\left[1-2x^{\frac{1}{2}}\right]$$

$$=0-2.\frac{1}{2}x^{\frac{1}{2}-1}=-x^{-\frac{1}{2}}=-\frac{1}{\sqrt{x}}$$

Product Rule:

If u and v are functions of x, ($u=f(x)$ and $v=g(x)$), then $\frac{d}{dx}(uv)=u\frac{d}{dx}(v)+v\frac{d}{dx}(u)$

Example:

: Find the differential coefficient $\frac{dy}{dx}$ of the following functions,

$(i) y=x^3\ln x$ $\qquad$ $(ii)\ y=x^2\cot^{-1}x$

Solution:

(i) Given that, $y=x^3\ln x$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \ln x)$$

$$= x^3 \frac{d}{dx}(lnx) + \ln x \frac{d}{dx}(x^3)$$

$$= x^3 . \frac{1}{x} + \ln x(2x^2)$$

$$\therefore \frac{dy}{dx} = x^2 + 2x^2$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(lnx) = \frac{1}{x}$$

$$\frac{d}{dx}(cot^{-1} x) = \frac{-1}{1+x^2}$$

(ii) Given that, $y = x^2 \cot^{-1} x$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \cot^{-1} x)$$

$$= x^2 \frac{d}{dx}(\cot^{-1} x) + \cot^{-1} x \frac{d}{dx}(x^2)$$

$$= x^2 \left(\frac{-1}{1+x^2}\right) + \cot^{-1} x(2x)$$

$$\therefore \frac{dy}{dx} = 2x \ \cot^{-1} x - \frac{x^2}{1+x^2}$$

Quotient Rule:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

Example:

Find the differential coefficient $\frac{dy}{dx}$ of the following functions,

(i) $y = \frac{2x^2+5}{3x-4}$ (ii) $y = \frac{\cos x}{1+\sin x}$

Solution:

(i) Given that, $y = \frac{2x^2+5}{3x-4}$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x^2+5}{3x-4}\right)$$

$$= \frac{(3x-4)\frac{d}{dx}(2x^2+5) - (2x^2+5)\frac{d}{dx}(3x-4)}{(3x-4)^2}$$

$$= \frac{(3x-4)4x - (2x^2+5)3}{(3x-4)^2}$$

$$= \frac{12x^2-16x-6x^2-15}{(3x-4)^2}$$

$$\therefore \frac{dy}{dx} = \frac{6x^2-16x-15}{(3x-4)^2}$$

(ii) Given that, $y = \frac{\cos x}{1+\sin x}$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1+\sin x}\right)$$

$$= \frac{(1+sin\ x)\frac{d}{dx}(cos\ x)-cos\ x\frac{d}{dx}(1+sin\ x)}{(1+sin\ x)^2}$$

$$= \frac{-\ sin\ x-sin^2 x-cos^2 x}{(1+sin\ x)^2}$$

$$= \frac{-\ sin\ x-(sin^2 x+cos^2 x)}{(1+sin\ x)^2}$$

$$= \frac{-sinx-1}{(1+\sin x)^2}$$

$$= \frac{-(sinx+1)}{(1+\sin x)^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{1+\sin x}$$

WEAK 7

Page:17-25

The Chain Rule:

If g is differentiable at x and f is differentiable at $g(x)$, then thecomposite function $F = fog$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \bullet g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

NOTE:

In using the Chain Rule we work from the outside to the inside. Formula 1 says that we differentiate the outer function f [at the inner function g(x)] and then we multiply by the derivative of the inner function

$$\frac{d}{dx}\underbrace{f}_{\substack{\text{outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

EXAMPLE:

Differentiate (a) $y = \sin(x^2)$ and (b) $y = sin^2 x$

SOLUTION:

(a) If $y = \sin(x^2)$, then the outer function is the sine function and the inner function is the squaring function, so the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{\sin}_{\substack{\text{outer}\\\text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{\cos}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \underbrace{(x^2)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{2x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

$$= 2x\cos(x^2)$$

(b) Note that $sin^2 x = (sinx)^2$. Here the outer function is the squaring function and the inner function is the sine function. So the Chain Rule gives

$$\frac{dy}{dx} = \frac{d}{dx} \underbrace{(\sin x)}_{\substack{\text{inner}\\\text{function}}}{}^2 = \underbrace{2}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}} \cdot \underbrace{(\sin x)}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{\cos x}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

Function as Power of another Function:

If u and v are functions of x, ($u = f(x)$ and $v = g(x)$), then $\frac{d}{dx}(u^v) = u^v \frac{d}{dx}\left(v\ ln(u)\right)$

Example:

Find the differential coefficient $\frac{dy}{dx}$ of the following functions

$(i) y = (sinx)^{ln\,x}$ $(ii)\ y = x^x + (sinx)^{\ln x}.$

Solution:

(i) Given that, $y = (sinx)^{ln\,x}$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}\{(\sin x)^{\ln x}\}$$

$$= (\sin x)^{\ln x}\frac{d}{dx}\{\ln x \ln(\sin x)\}$$

$$= (sinx)^{\ln x}\left[\ln x\ \frac{d}{dx}\{ln(\sin x)\} + \ln(\sin x).\frac{d}{dx}(\ln x)\right]$$

$$= (\sin x)^{\ln x}\left[\ln x.\frac{1}{x}.\cos x + \ln(\sin x).\frac{1}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\sin x)^{\ln x}\left[\ln x.\frac{1}{x}.\cos x + \ln(\sin x).\frac{1}{x}\right]$$

(ii)

Given that, $y = x^x + (sinx)^{\ln x}$

Differentiating with respect to x then we get,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{x^x + (\sin x)^{\ln x}\right\}$$

$$= \frac{d}{dx}(\mathrm{x^x}) + \frac{d}{dx}\left\{(\sin x)^{\ln x}\right\}$$

$$= x^x \frac{d}{dx}(x \ln x) + (\sin x)^{\ln x} \frac{d}{dx}\{\ln x \ln(\sin x)\}$$

$$= x^x (x.\frac{1}{x} +$$

$$\ln x) + (\sin x)^{\ln x}\left\{\ln x.\frac{1}{\sin x}.\cos x + \frac{\ln(sinx)}{x}\right\}.$$

$$\therefore \frac{dy}{dx} = x^x(1 + \ln x) + (\sin x)^{\ln x}\left\{\ln x.\cot x + \frac{\ln(\sin x)}{x}\right\}$$

Differentiation Parametric Equations:

If in the equation of a curve $y = f(x), x$ and y are expressed in terms of a third variable known as parameter that is, $x = \phi(t), y = \psi(t)$ then the equations are called a parametric equation.

If both x and y are function of t then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Example:

Find the differential coefficient $\frac{dy}{dx}$ of the following functions:

(i) x = a(t + sin t) , y = a(1 − cos t)

(ii) x = a(cos t + t sin t) , y = a(sin t − t cos t)

Solution:

(i) Given that,

$x = a(t + \sin t)$ ……… (1)

and $y = a(1 - \cos t)$ ……… (2)

Differentiating (1) and (2) with respect to t so we get,

$\frac{dx}{dt}$ = a(1 + cos t)

and $\frac{dy}{dt} = a \sin t$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \sin t}{a\,(1+cost)}$

$= \frac{2\, sin\frac{t}{2} cos\frac{t}{2}}{2cos^2\frac{t}{2}}$

$\therefore \frac{dy}{dx} = tan\,\frac{t}{2}$

(ii) Given that,

$$x = a(cos\,t + t\,sin\,t) \ldots\ldots\ldots.\ldots \;(1)$$

and $y = a(sin\,t - t\,cos\,t) \ldots\ldots\ldots..\ldots \;(2)$

Differentiating (1) and (2) with respect to t so we get,

$$\frac{dx}{dt} = a(-sin\,t + t\,cos\,t + sin\,t)$$

$$= a\,t\,cos\,t$$

and $\frac{dy}{dt} = a(cos\,t + t\,sin\,t - cos\,t)$

$$= a\,t\,sin\,t$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$= \frac{at\,sint}{at\,cost}$$

$$\therefore \frac{dy}{dx} = \tan t$$

Implicit Function:

When the relationship between x and y is expressed by an equation of the form $f(x,y) = 0$, it is often very difficult or even impossible to express y as a function of x. In such cases, y is called an Implicit function of x.

Example:

(i) $x^2 + y^2 = a^2$

(ii) $x^2 + xy + y^2 = 4$

Process of Implicit Differentiation:

Step 1: Differentiate both sides of the equation with respect to x

Step 2: Treat y as a function of x, so $\frac{\mathrm{d}}{\mathrm{dx}}(\mathrm{y}) = \frac{dy}{dx}$

Step 3: Solve for $\frac{dy}{dx}$ after differentiation

Example 1:

If $x^p y^q = (x+y)^{p+q}$ then find $\frac{dy}{dx}$

Solution:

Given that $x^p y^q = (x+y)^{p+q}$

Taking ln in both side we get

$$plnx + qln\, y = (p+q)\, ln(x+y)$$

Differentiating both sides with respect to x, we get

$$\frac{p}{x} + \frac{q}{y} . \frac{dy}{dx} = \frac{p+q}{x+y}\left(1 + \frac{dy}{dx}\right) or \left[\frac{q}{y} - \frac{p+q}{x+y}\right]\frac{dy}{dx} = \frac{p+q}{x+y} - \frac{p}{x}$$

or, $(qx + qy - py - qy)\frac{dy}{dx} = pz + qx - px - py$

or, $x(qx - py)\frac{dy}{dx} = y(qx - py)$

or, $\frac{dy}{dx} = \frac{y}{x}.$

Example 2:

Find the differential coefficient $\frac{dy}{dx}$ of the function $x^y + y^x = a^b$

Solution:

Given that $x^y + y^x = a^b$

Differentiating both sides with respect to x, we get

$$x^y \frac{d}{dx}(y \ln x) + y^x \frac{d}{dx}(x \ln y) = 0$$

or, $x^y(\frac{y}{x} + \frac{dy}{dx}. \ln x) + y^x(\frac{x}{y}.\frac{dy}{dx} + \ln y\,) = 0$

or, $(yx^{y-1} + y^x \ln y) + (x^y \ln x + xy^{x-1})\frac{dy}{dx} = 0$

$$\therefore \frac{dy}{dx} = -\frac{yx^{y-1} + y^x \ln y}{x^y \ln x + xy^{x-1}}$$

WEAK 8

Page:17-25

Successive Derivative:

If $y = f(x)$ be a function of x, then the first order derivative of y with respect to x is denoted by

$$\frac{dy}{dx}, \quad f'(x), \quad y_{1,}, \quad y^{(1)}, \quad f^{(1)}(x) \text{ etc.}$$

Again, the derivative of the first order derivative of y with respect to x is called the second order derivative and is denoted by

$$\frac{d^2y}{dx^2}, \quad f'(x), \quad y_2\,, \quad y^{(2)}, \quad f^{(2)}(x) \text{ etc.}$$

Similarly, the nth derivative of y with respect to x is denoted by

$$\frac{d^ny}{dx^n}, \quad f^n(x), \quad y_n, \quad y^{(n)}, \quad f^{(n)}(x) \text{ etc.}$$

Example: If $y = 3x^4 - 4x^3 + 3x^2 + 2x + 5$ then find $y', \quad y'', \quad y''', \quad y^{(4)}$

Solution: Given that the function is $y = 3x^4 - 4x^3 + 3x^2 + 2x + 5$

Then, $y' = \frac{d}{dx}(3x^4 - 4x^3 + 3x^2 + 2x + 5)$

$= 12x^3 - 12x^2 + 6x + 2$

$y'' = \frac{d}{dx}(y') = \frac{d}{dx}(12x^3 - 12x^2 + 6x + 2)$

$= 36x^2 - 24x + 6$

$y''' = \frac{d}{dx}(y'') = \frac{d}{dx}(36x^2 - 24x + 6)$

$= 72x - 24$

$y^{(4)} = \frac{d}{dx}(y''') = \frac{d}{dx}(72x - 24)$

$= 72$

Example 1: Find the differential coefficient $\frac{dy}{dx}$ of the following functions:

(i) $y = x^n$ (ii) $y = e^{ax}\, y$ (iii) $y = cos(ax + b)$

Solution: (i) Given that, $y = x^n$

Differentiating with respect to x so we get,

$$y_1 = nx^{n-1}$$

$$y_2 = n(n-1)x^{n-2}$$

$$y_3 = n(n-1)(n-2)x^{n-3}$$

Similarly,

$$y_n = n(n-1)(n-2)\cdots[n-(n-1)]x^{n-n}$$

$$= n(n-1)(n-2)\cdots 3.2.1$$

$$= n!$$

(ii) Given that, $y = e^{ax}$

Differentiating with respect to x

so we get,

$$y_1 = ae^{ax}$$

$$y_2 = a^2e^{ax}$$

$$y_3 = a^3e^{ax}$$

Similarly,

$$y_n = a^n e^{ax}$$

(iii) Given that, $y = cos(ax + b)$

Differentiating with respect to x so we get,

$$y_1 = -asin(ax + b)$$

$$= \text{acos}\left[\frac{\pi}{2} + (ax + b)\right]$$

$$y_2 = -a^2sin\left[\frac{\pi}{2} + (ax + b)\right]$$

$$= a^2cos\left[2\frac{\pi}{2} + (ax + b)\right]$$

$$y_3 = -a^3sin\left[\frac{\pi}{2} + (ax + b)\right]$$

$$= a^3cos\left[3\frac{\pi}{2} + (ax + b)\right]$$

Similarly,

$$y_n = a^n cos\left[n\frac{\pi}{2} + (ax + b)\right]$$

LEIBNITZ'S THEOREM:

If u and v are any two functions of x such that all their desired differential coefficients exist, then the n-th differential coefficient of their product is given by

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + {}^n c_1 \frac{d^{n-1}u}{dx^{n-1}} \frac{dv}{dx} + {}^n c_2 \frac{d^{n-2}u}{dx^{n-2}} \frac{d^2 v}{dx^2} + \cdots\cdots + {}^n c_r \frac{d^{n-r}u}{dx^{n-r}} \frac{d^r v}{dx^r} + \cdots\cdots + u \frac{d^n v}{dx^n}$$

or,

$$\frac{d^n}{dx^n}(uv) = \frac{d^n u}{dx^n} v + n \frac{d^{n-1}u}{dx^{n-1}} \frac{dv}{dx} + \frac{n(n-1)}{2!} \frac{d^{n-2}u}{dx^{n-2}} \frac{d^2 v}{dx^2} + \cdots\cdots + \frac{n!}{r!\,(n-r)!} \frac{d^{n-r}u}{dx^{n-r}} \frac{d^r v}{dx^r} + \cdots\cdots + u \frac{d^n v}{dx^n}$$

Example 1:

If $= sin(m\, sin^{-1} x)$ then show that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and deduced that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Solution:

Given $y = sin(m\, sin^{-1} x)$

Differentiating both sides with respect to x, we get

$$y_1 = cos(m\, sin^{-1} x)\frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2){y^2}_1 = m^2 cos^2(m\, sin^{-1} x)$$ [squaring both sides]

$$\Rightarrow (1-x^2){y^2}_1 = m^2 - m^2 sin^2(m\, sin^{-1} x) = m^2 - m^2y^2$$

$$\Rightarrow (1-x^2){y^2}_1 + m^2y^2 = m^2$$

Now again differentiating both sides, we get

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 + 2m^2yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$$

(showed)

Again differentiating both sides in times by Leibnitz's theorem

$$\frac{d^n}{dx^n}\{(1-x^2)y_2\} - \frac{d^n}{dx^n}(xy_1) + m^2\frac{d^n}{dx^n}(y) = 0$$

$$\Rightarrow (1-x^2)\frac{d^n}{dx^n}(y_2) + n(-2x)\frac{d^{n-1}}{dx^{n-1}}(y_2) + \frac{n(n-1)}{2}(-2)\frac{d^{n-2}}{dx^{n-2}}(y_2) - x\frac{d^n}{dx^n}(y_1) - n\frac{d^{n-1}}{dx^{n-1}}(y_1) + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}2y_n - xy_{n+1} - ny_n + m^2y_n = 0$$

$$\Rightarrow \ (1-x^2)y_{n+2} - (2n-1)xy_{n+1} - (n^2-m^2)y_n = 0$$

(showed)

Example 2:

If $y = a\,cos(ln\,x) + b\,sin(ln\,x)$ where and are constant, then show that $(1-x^2)y_2 - xy_1 + m^2y = 0$ and deduced that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

Solution:

Given $y = sin(m\,sin^{-1}x)$

Differentiating both sides with respect to x, we get

$$y_1 = cos(m\,sin^{-1}x)\frac{m}{\sqrt{1-x^2}} \quad \therefore$$

or, $(1-x^2){y^2}_1 = m^2\,cos^2(m\,sin^{-1}x)$ [squaring both sides]

or, $(1-x^2){y^2}_1 = m^2 - m^2\,sin^2(m\,sin^{-1}x) = m^2 - m^2y^2$

or, $(1-x^2){y^2}_1 + m^2y^2 = m^2$

Now again differentiating both sides, we get

$$(1-x^2)2y_1y_2 + (-2x)y_1^2 + 2m^2yy_1 = 0$$

$\therefore (1-x^2)y_2 - xy_1 + m^2y = 0$ (Showed)

Again differentiating both sides in times by Leibnitz's theorem,

or, $\frac{d^n}{dx^n}\{(1-x^2)y_2\} - \frac{d^n}{dx^n}(xy_1) + m^2\frac{d^n}{dx^n}(y) = 0$

or, $(1-x^2)\frac{d^n}{dx^n}(y_2) + n(-2x)\frac{d^{n-1}}{dx^{n-1}}(y_2) + \frac{n(n-1)}{2}(-2)\frac{d^{n-2}}{dx^{n-2}}(y_2) - x\frac{d^n}{dx^n}(y_1) - n\frac{d^{n-1}}{dx^{n-1}}(y_1) + m^2 y_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{n(n-1)}{2}2y_n - xy_{n+1} - ny_n + m^2 y_n = 0$

or, $(1-x^2)y_{n+2} - (2n-1)xy_{n+1} - (n^2 - m^2)y_n = 0$

(Showed)

Exercises:

(i) If $\ln y = \tan^{-1} x$ then show that, $(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

(ii) If $y = \sin\{a \ln(x+b)\}$ then show that ,

(1) $(x+b)^2 y_2 + (x+b)y_1 + a^2 y = 0$

(2) $(x+b)^2 y_{n+2} + (2n+1)(x+b)y_{n+1} + (n^2+a^2)y_n = 0$

(iii) If $y = e^{m\cos^{-1} x}$ then, proved that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

(iv) If $y = \left(x + \sqrt{x^2+1}\right)^m$ then proved that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2+m^2)y_n = 0$

(v) If $y = \ln\left(x + \sqrt{x^2+a^2}\right)$ then show that, $(x^2+a^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$

WEAK 9

Page:17-25

Velocity is the differentiation of displacement with respect to time .

$$\therefore \textbf{Velocity } v = \frac{ds}{dt}$$

Acceleration is the differentiation of velocity with respect to time.

$$\therefore \textbf{Acceleration } a = \frac{dv}{dt} = \frac{d^2}{dt^2}$$

Question:

If the displacement of a particle moving along a straight line at time t is expressed by $s = 63t - 6t^2 - t^3$, then after $2s$, find (i) displacement, (il) velocity, (iii) acceleration.

Solution:

Given, $s = 63t - 6t^2 - t^3 \ldots\ldots\ldots$ (i)

Putting $t = 2$ at (i)

$\Rightarrow s = 63 \times 2 - 6 \times 2^2 - (2)^3$

$\Rightarrow s = 126 - 24 - 8$

$\therefore s = 94\ m$

$\therefore$ After $2s$, the particle's displacement = 94 m

Again, the rate of change of displacement with respect to time is velocity.

$\therefore$Differentiating both sides of (i) with respect to t,

$$\frac{ds}{dt} = \frac{d}{dt}(63t - 6t^2 - t^3)$$

$$\Rightarrow v = \frac{d}{dt}(63t) - \frac{d}{dt}(6t^2) - \frac{d}{dt}(t^3) \quad [\because \text{velocity, } v = \frac{ds}{dt}]$$

$$\Rightarrow v = 63 - 12t - 3t^2 \ldots\ldots\ldots \text{(ii)}$$

Placing $t = 2$ in (ii), we get

$$v = 63 - 12 \times 2 - 3 \times 2^2$$

$$\Rightarrow v = 27 \text{ m/s}$$

$\therefore$ After $2s$, the velocity of the particle is 27 m/s.

$\therefore$ Differentiating both sides of (ii) with respect to t,

$$\frac{dv}{dt} = \frac{d}{dt}(63 - 12t - 3t^2)$$

$$\Rightarrow a = \frac{d}{dt}(63) - \frac{d}{dt}(12t) - \frac{d}{dt}(3t^2) \ [\because \ a = \frac{dv}{dt}]$$

$$\Rightarrow a = 0 - 12.\frac{dt}{dt} - 3.\frac{d}{dt}(t^2)$$

$$\Rightarrow a = -12 - 6t$$

$$\therefore a = -(12 + 6t) \ldots \ldots \ldots \text{(iii)}$$

Placing $t = 2$ in (iii), $a = -(12 + 12)$

$\therefore a = -24\ m/s^2$ (Ans.)

Critical number:

A critical number of a function f is a number c in the domain off such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Question:

Find the critical numbers of (a) $f(x) = x^3 - 3x^2 + 1$ and (b) $f(x)=x^{\frac{3}{5}}(4 - x)$.

Solution:

(a) Given that $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x = 3x(x - 2)$.

Since $f'(x)$ exists for all x, the only critical numbers of f occur when $f'(x) = 0$,

that is, $3x(x - 2) = 0$

$\Rightarrow x = 0$ or $x = 2$.

Thus the critical numbers are 0 and 2.

(b) Given that $f(x) = x^{\frac{3}{5}}(4 - x)$

$$\therefore\ f'(x) = x^{\frac{3}{5}}(-1) + (4 - x)\left(\frac{3}{5}x^{\frac{-2}{5}}\right)$$

$$= -\ x^{\frac{3}{5}} + \frac{3\,(4-x)}{5\,x^{2/5}}$$

$$= \frac{-\ 5x + 3\,(4-x)}{5\,x^{2/5}}$$

$$= \frac{12 - 8x}{5x^{2/5}}$$

For critical point $f\,'(x) = 0$

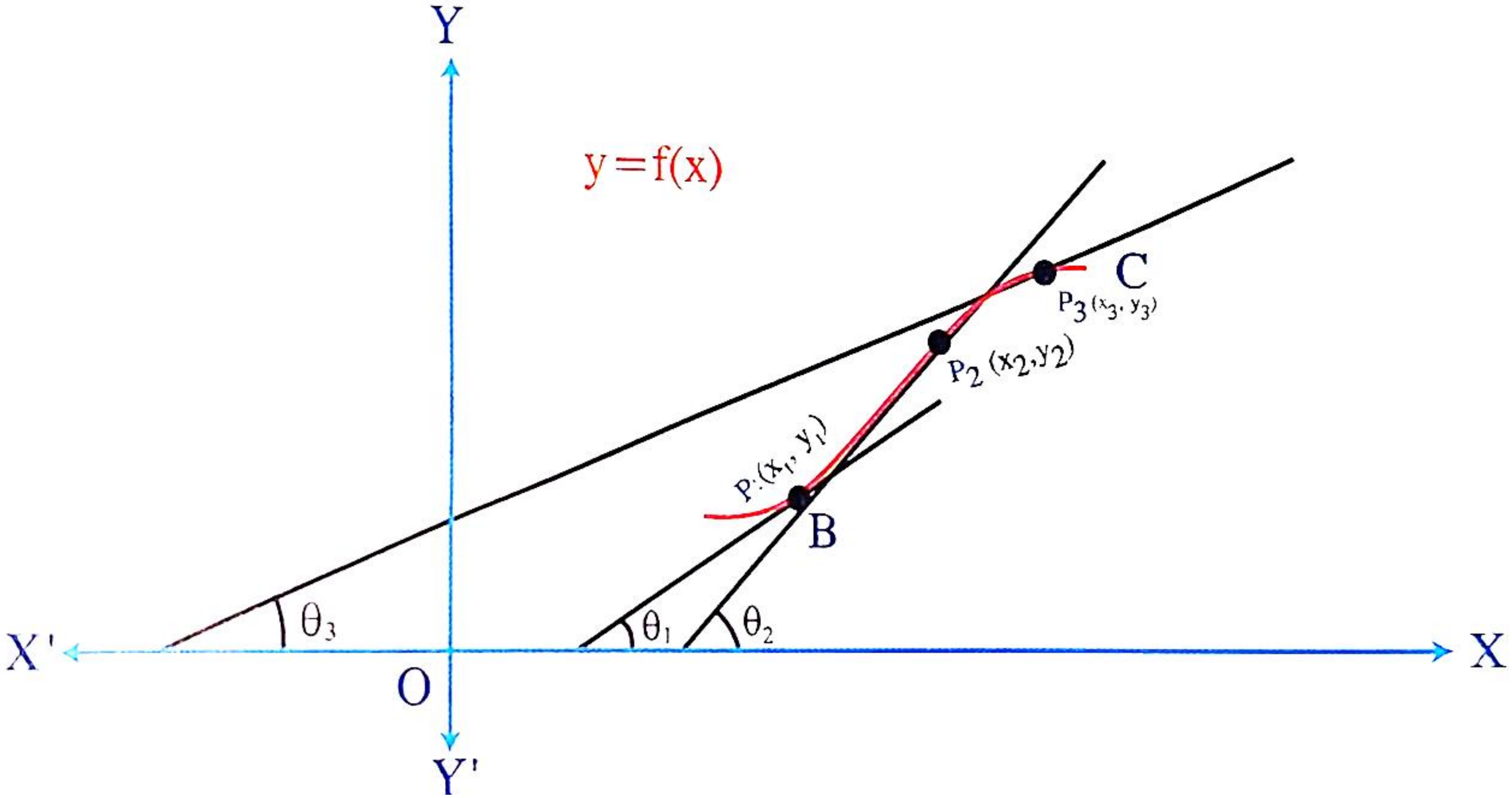
that is, $12 - 8x = 0$

$$\Rightarrow x = \frac{3}{2}$$

and $f\,'(x)$ does not exist when $x = 0$.

Thus the critical numbers are $\frac{3}{2}$ and 0.

Increasing Function:



The function is increasing in the region BC. If in BC region, any three points $P_1(x_1, y_1)$,$P_2(x_2, y_2)$ & $P_3(x_3, y_3)$ are taken and tangents are drawn, then the tangents make acute angles θ_1, θ_2 & θ_3 with the positive direction of x −axis respectively. That is, the slopes of these tangents $tan\theta_1$, $tan\theta_2$ & $tan\theta_3$ are all positive.

That is, we can say, the value of $\frac{dy}{dx}$ will be positive at points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ & $P_3(x_3, y_3)$. That is, $\frac{dy}{dx}\,|_{x=x_1} > 0$, $\frac{dy}{dx}\,|_{x=x_2} > 0$ & $\frac{dy}{dx}\,|_{x=x_3} > 0$

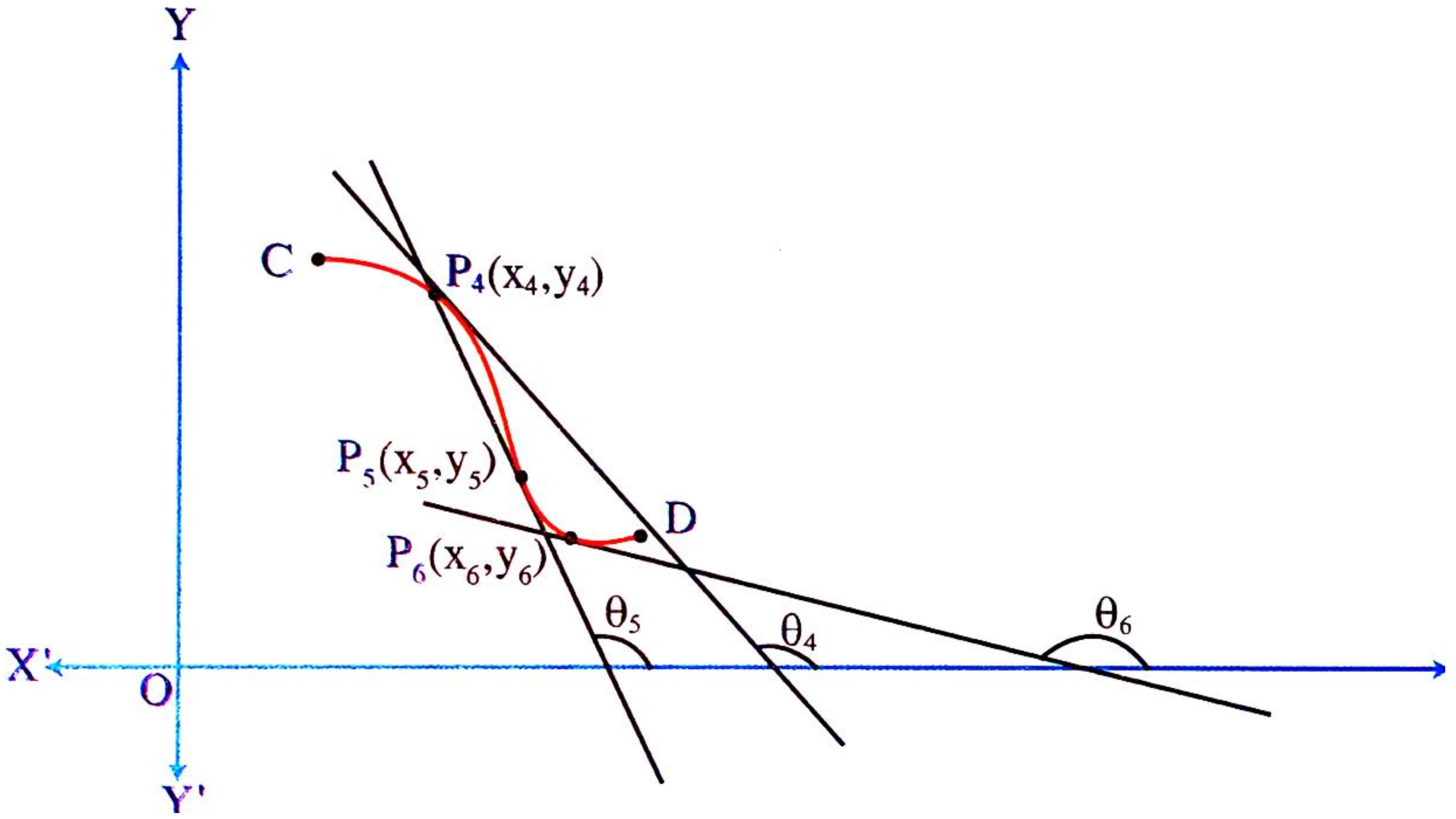
That is, the condition for an increasing function is $\frac{dy}{dx} > 0$.

Note:

If within an interval, with the increase of x, value of $f(x)$ increases, that is, $f\,'(x) > 0$, then $f(x)$ is called an increasing function within that interval. That is, if $\frac{dy}{dx}> 0$, then the function is increasing. The tangent at any dx point of the increasing function makes acute angle with the positive direction of x-axis.

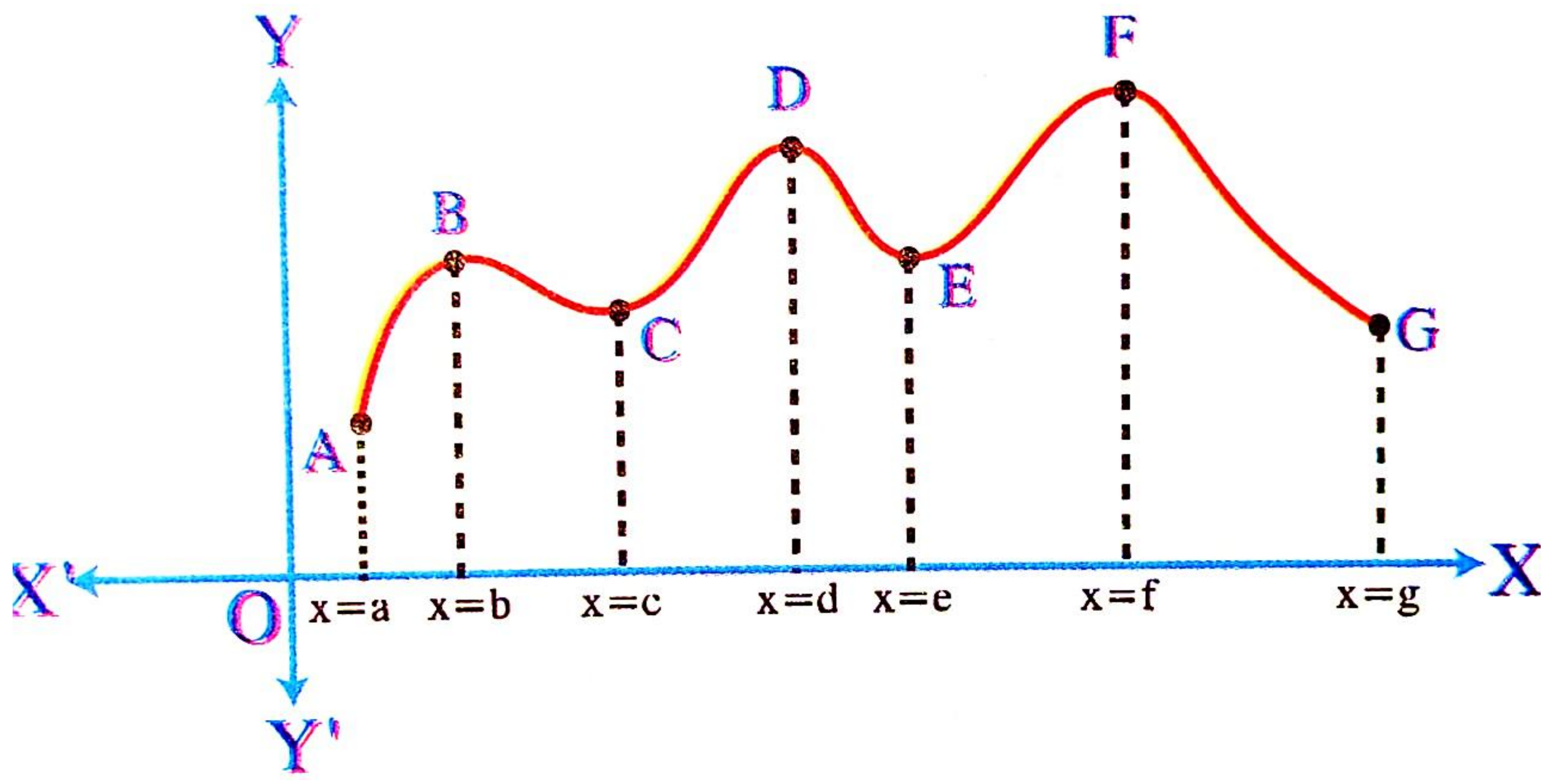
If within any interval, with the Increase of x, the value of $f(x)$ decreases, that is, $f'\ (x) < 0$ then the function $f(x)$ is called a decreasing function within that interval. That is, if $\frac{dy}{dx} < 0$ then the function is decreasing. The tangent at any point of the decreasing function makes an obtuse angle with the positive direction of x-axis

Decreasing Function:



The Function is decreasing in region CD in the figure. Now taking any three points $P_4(x_4, y_4)$,$P_5(x_5, y_5)$ & $P_6(x_6, y_6)$. in the region CD, it can be seen that the tangent drawn at these points to the positive direction of x-axis produces angles θ_4,θ_5 & θ_6 respectively, each of which is an obtuse angle. Hence, the slope of the tangent will be negative. That is, $\frac{dy}{dx} < 0$ at points $P_4(x_4, y_4)$,$P_5(x_5, y_5)$ & $P_6(x_6, y_6)$.

That is, in case of decreasing function, the condition is $\frac{dy}{dx} < 0$



The graph of the above function with respect to x −axis:

$(a, b) \rightarrow$ increasing $\rightarrow \frac{dy}{dx} > 0$

$(b, c) \rightarrow$ decreasing$\rightarrow \frac{dy}{dx} < 0$

$(c, d) \rightarrow$ increasing $\rightarrow \frac{dy}{dx} > 0$

$(d, e) \rightarrow$ decreasing$\rightarrow \frac{dy}{dx} < 0$

(e, f)$\rightarrow$ increasing $\rightarrow \frac{dy}{dx} > 0$

(f, g)$\rightarrow$ decreasing $\rightarrow \frac{dy}{dx} < 0$

Question:

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing

and where it is decreasing.

Solution:

Given that $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$\therefore f'(x) = 12x^3 - 12x^2 - 24x$$

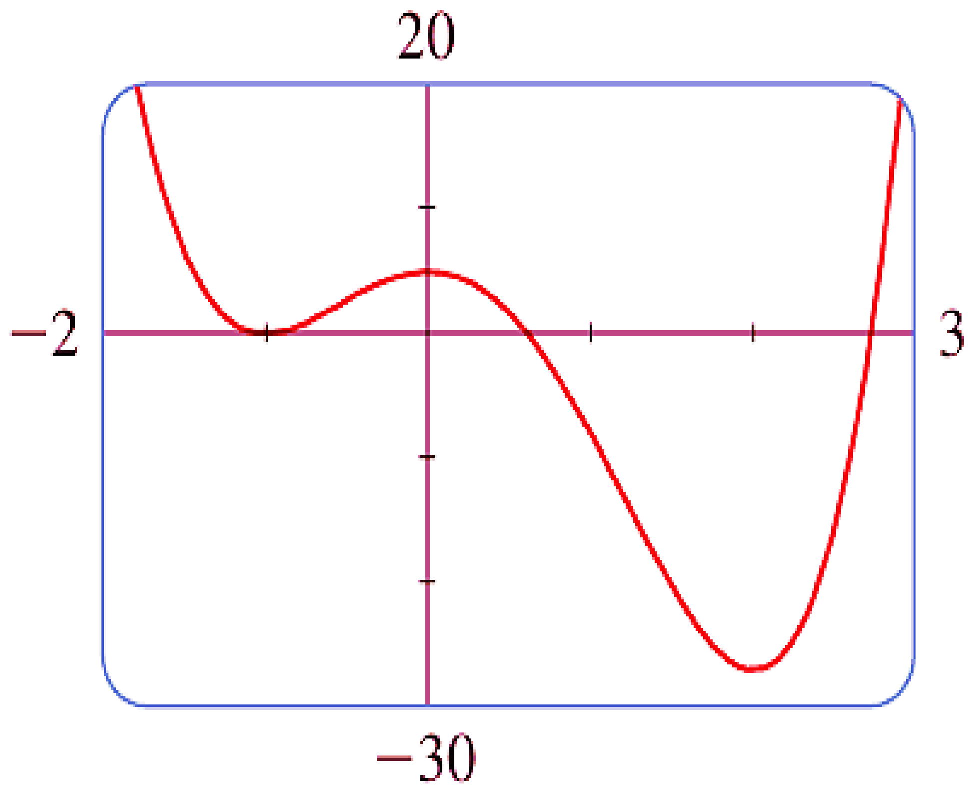
$$= 12x(x - 2)(x + 1)$$

For critical point $f'(x) = 0$

$$\therefore 12x(x - 2)(x + 1) = 0$$

$$\Rightarrow x = 0, -1, 2$$

Now, $x = 0, -1, 2$ divides all real numbers into $x < -1, -1 < x < 0, 0 < x < 2$ & $x > 2$ intervals.



Interval	Sign of $12x$	Sign of $x - 2$	Sign of $x + 1$	Sign of $f'(x)$	Comment
$x < -1$	$-$	$-$	$-$	$-$	Decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$-$	$+$	$+$	Increasing on $(-1, 0)$
$0 < x < 2$	$+$	$-$	$+$	$-$	Decreasing on $(0, 2)$
$x < 2$	$+$	$+$	$+$	$+$	Increasing on $(2, \infty)$

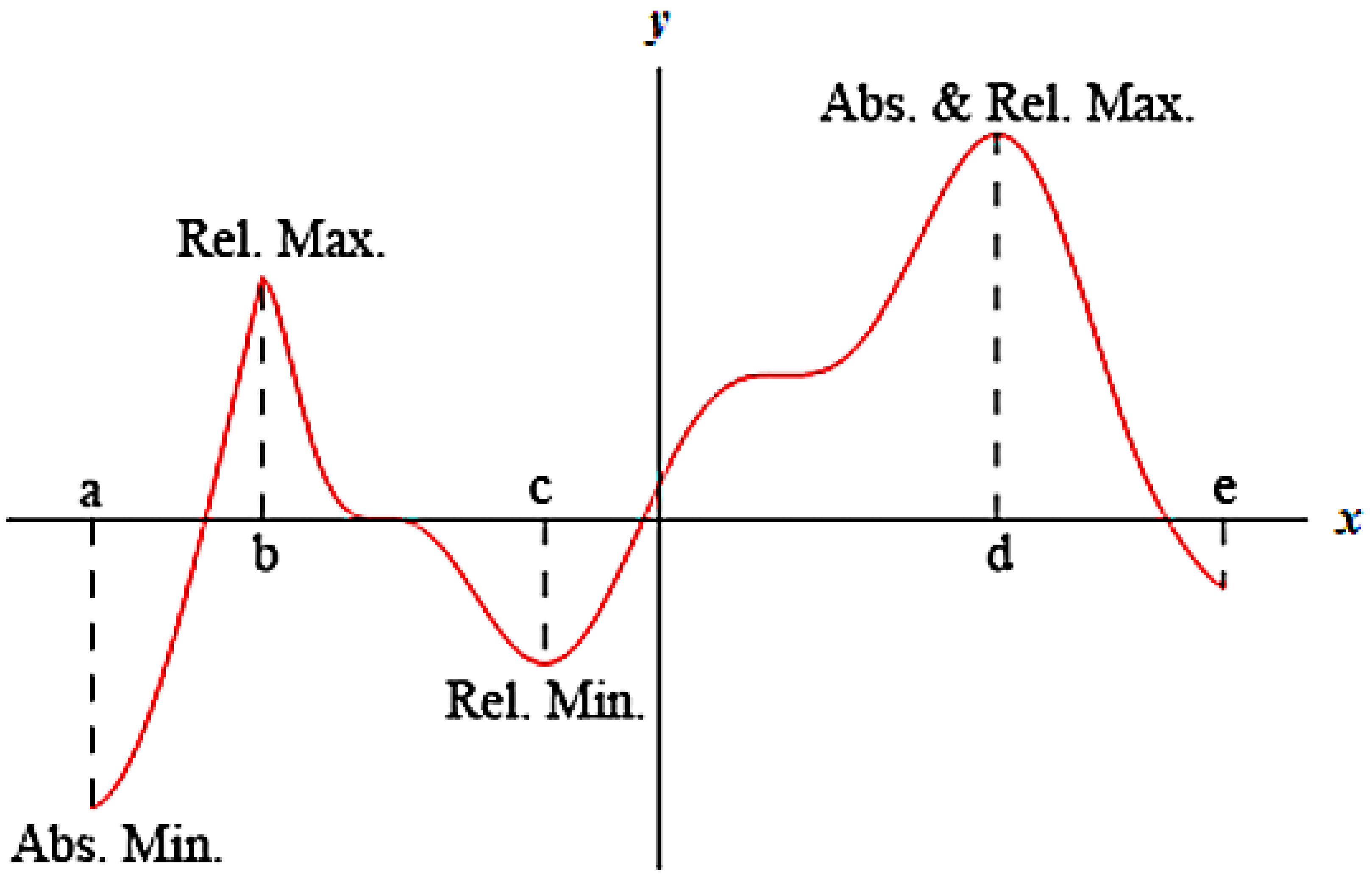
WEAK 10

Page:17-25

Maximum and Minimum:

A function $f(x)$ is said to have a maximum for a value a of x if $f(a)$ is greater than any other value that the function can have in the small neighborhood of $x = a$.

Similarly, a function $f(x)$ said to have a minimum for a value a of x if $f(a)$ is less than any other value that the function can have in the small neighborhood of $x = a$.

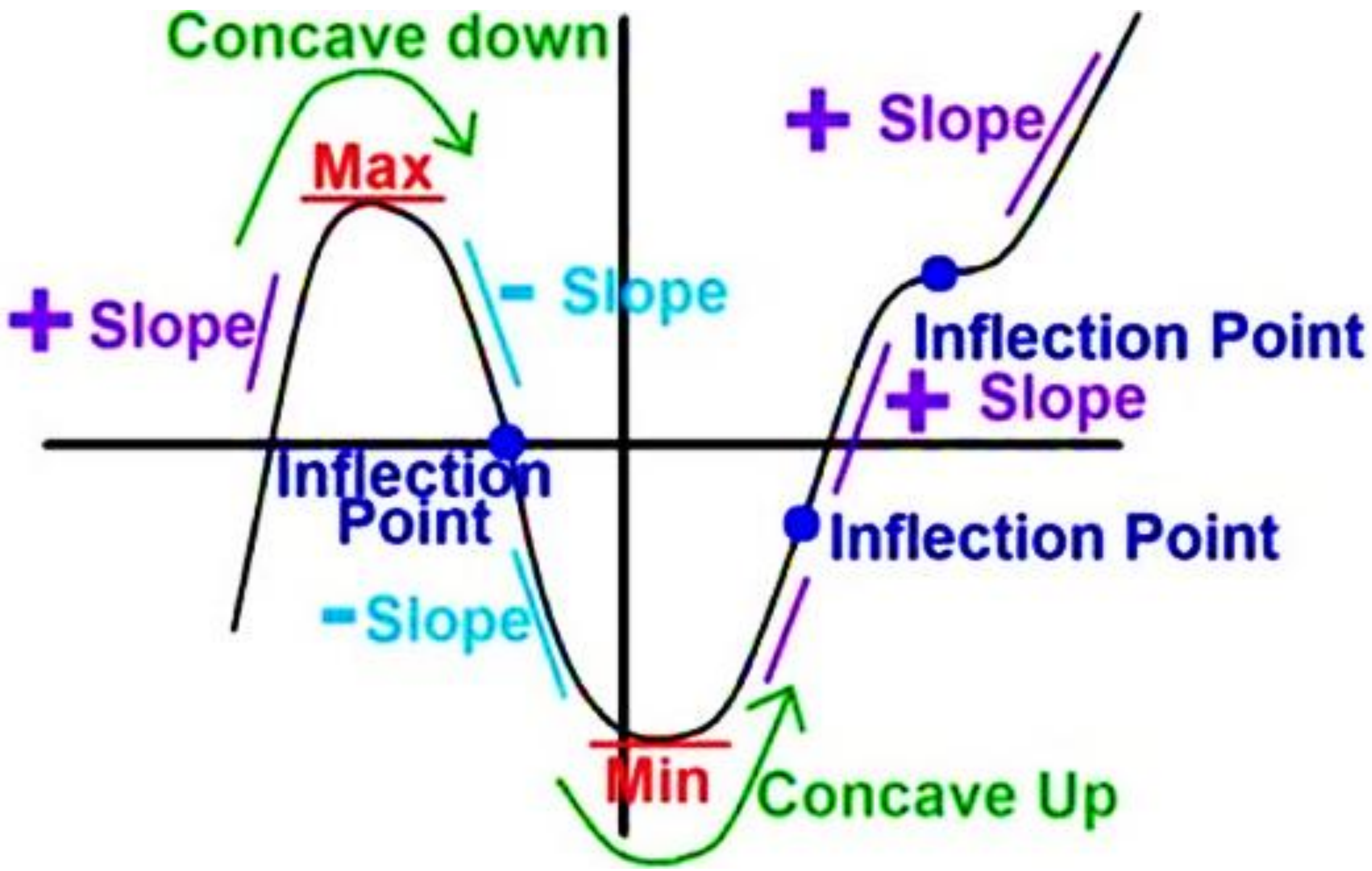


Critical Points:

The points on the graph of the function $y = f(x)$ at which the slope of the tangent line is zero are called Critical Points. The Critical points are obtained by solving $\frac{dy}{dx} = 0$ i. e. $f'(x) = 0$.

Inflection Points:

The points at which the graph of the function $y = f(x)$ changes its concavity are called Inflection points. The Inflection points are obtained by solving $\frac{d^2y}{dx^2} = 0$ or $f''(x) = 0$.



For the function $y = f(x)$ we may say that at the point $x = c$

$x = c$	Test condition	Decision
If $f'(c) = 0$	$f''(c) < 0$	$f(x)$ has maximum value at c
If $f'(c) = 0$	$f''(c) > 0$	$f(x)$ has minimum value at c
If $f'(c) = 0$	$f''(c) = 0$	Test is inconclusive

Question:

Find the Critical points, point of Inflection, maximum and minimum values of the following functions:

(a) $f(x) = 4x^3 - 9x^2 + 6x$ (b) $f(x) = 2x^3 - 3x^2 - 12x$ (c) $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution:

(a) Given $f(x) = 4x^3 - 9x^2 + 6x$

$$\therefore f'(x) = 12x^2 - 18x + 6 \text{ and } f''(x) = 24x - 18$$

For maximum and minimum values (that is for critical points), $f'(x) = 0$

$$\Rightarrow 12x^2 - 18x + 6 = 0$$

$$\Rightarrow 2x^2 - 3x + 1 = 0$$

$$\Rightarrow (2x - 1)(x - 1) = 0$$

$$\Rightarrow x = \frac{1}{2}, 1$$

At $x = \frac{1}{2}$, $f''\left(\frac{1}{2}\right) = 24 \times \frac{1}{2} - 18 = -6 < 0$

Therefore $f(x)$ has a maximum value at $x = \frac{1}{2}$.

So, the maximum value f(x) at $x = \frac{1}{2}$ is,

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 9\left(\frac{1}{2}\right)^2 + 6\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{9}{4} + 3 = \frac{4 - 18 + 24}{8} = \frac{5}{4}$$

Again at $x = 1$, $f''(1) = 24 \times 1 - 18 = 6 > 0$

Therefore $f(x)$ has a minimum value at $x = 1$

So, the minimum value $f(x)$ at x = 1 is, $f\ (1) = 4(1)^2 - 9(1)^2 + 6(1) = 1$

Again, Inflection points are obtained by solving $f''\ (x) = 0 \Rightarrow 24x - 18 = 0 \Rightarrow x = \frac{3}{4}$

Therefore, $f\left(\frac{3}{4}\right) = 4(\frac{3}{4})^3 - 9\left(\frac{3}{4}\right)^2 + 6\left(\frac{3}{4}\right) = 4\frac{27}{81} - 9\frac{9}{16} + \frac{18}{4} = \frac{37}{48}$

∴Point of inflection is $\left(\frac{3}{4}, \frac{37}{48}\right)$

(b) Given that $f(x) = 2x^3 - 3x^2 - 12x$

$\therefore f'(x) = 6x^2 - 6x - 12$ and $f''(x) = 12x - 6$

For maximum and minimum values (that is for critical points) $f'(x) = 0$

$$\Rightarrow 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)\,(x + 1) \;=\; 0$$

$$\Rightarrow x = 2, -1$$

At $x = -1,\ f''(2) = -12 - 6 = -18 < 0$

Therefore $f(x)$has a maximum value at $x = -1$

So, the maximum value $f(x)$ at $x\ = -1$ is,$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) = 7$.

Again at $x =\ 2,\ f''(2) = 24 - 6 = 18 > 0$

Therefore $f(x)$ has a maximum value at $x = 2$

So, the maximum value $f(x)$ at $x = 2$ is, $f(2) = 2(2)^3 - 3(2)^2 - 12(2) = -20.$

Again, inflection points are obtained by solving, $f''(x) = 0$

$$\Rightarrow 12x - 6 = 0$$

$$\Rightarrow \mathrm{x} = \frac{1}{2}$$

Therefore $f\left(\frac{1}{2}\right) = 2(\frac{1}{2})^3 - 3\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{3}{4} - 6 = -\frac{13}{2}$

$\therefore$Point of inflection is $\left(\frac{3}{4}, \frac{37}{48}\right)$

(c) Given, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$\Rightarrow f''(x) = 20x^3 - 60x^2 + 30x$$

$$\Rightarrow f'''(x) = 60x^2 - 120x + 30$$

For maximum and minimum values (that is for critical points), $f'(x) = 0$

$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow 5x^2(x^2 - 3x - x + 3) = 0$$

$$\Rightarrow x = 0,1,3$$

Now $f''(0) = 0$ and $f'''(0) = 30 \neq 0$,

Therefore $f(x)$ has no maximum or minimum values at $x = 0$, so $x = 0$ is the saddle point of $f(x)$

Again, at $x = 1$, $f''(1) = 20 \times 1^3 - 60 \times 1^2 + 30 \times 1 = -10 < 0$.

So, $f(x)$ has a maximum or minimum value at $x = 1$.

So, the maximum value $f(x)$ at $x = 1$ is $f(1) = 1 - 5 + 5 - 1 = 0$.

Again, at $x = 3$, $f''(3) = 20 \times 3^3 - 60 \times 3^2 + 30 \times 3 = 90 > 0$

Therefore $f(x)$ has a minimum value at $x = 3$.

So, the minimum value $f(x)$ at $x = 3$ is $f(3) = 3^5 - 5.3^4 + 5.3^3 - 1 = -28$

Question:

An open box is to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

Solution:

Let x be the length of the square to be cut out and V be the volume of the resulting box. As we are removing a square of side x from each corner, the resulting box will have dimensions $16 - 2x$ by $30 - 2x$ by x.

Since the volume of a box is the product of its dimensions, we have

$$V = (16 - 2x)(30 - 2x)\ x$$

$$= 4x(8 - x)(15 - x)$$

$$= 480x - 92x^2 + 4x^3$$

Now $\frac{dv}{dx} = 480 - 184x + 12x^2$ and $\frac{d^2v}{dx^2} = -184 + 24x$

For maximum value $\frac{dv}{dx} = 0$

$$\Rightarrow 480 - 184x + 12x^2 = 0$$

$$\Rightarrow 4(x - 12)(3x - 10)$$

$$\Rightarrow \ (x - 12)\ (3x - 10) = 0$$

$$\therefore x = 12 \text{ and } x = \frac{10}{3}$$

As x represents a length, it cannot be negative and because the width of the cardboard is 16 inches, we cannot cutout squares whose sides are more than 8 inches long. So, $x = \frac{10}{3}$

At $x = \frac{10}{3}$, $\frac{d^2v}{dx^2} = -184 + 24 \times \frac{10}{3} = -104 < 0$

Therefore, maximum volume occurs when $x = \frac{10}{3}$ inches and

maximum volume $= \left(16 - 2 \times \frac{10}{3}\right)\left(32 - 2 \times \frac{10}{3}\right) \times \frac{10}{3} = \frac{19600}{27}$

Question:

A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

Solution:

Let x be the length, y be the width and A be the area of the rectangular garden.

Then $A = xy$.................(i)

Since the perimeter of the rectangular garden is 100 ft.,

$\therefore\ 2x + 2y = 100 \Rightarrow y = 50 - x$ ....(ii)

Substituting (ii) in (i)

$$A = x(50 - x) = 50x - x^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (iii)$$

Differentiating (ii) with respect to x we have $\frac{dA}{dx} = 50 - 2x$ And $\frac{d^2A}{dx^2} = -2$

Setting $\frac{dA}{dx} = 0$ we obtain, $50 - 2x = 0$ or, $x = 25$.

Since $\frac{d^2A}{dx^2} < 0$, Thus the maximum occurs at one of the values $x = 25$

$\therefore$ Maximum area= $25(50 - 25) = 625$. Thus, the maximum area is 625 ft^2 occurs at $x = 25$

Substituting $x = 25$ in (ii) $y = 50 - 25 = 25$

So the rectangle of perimeter 100ft with greater area is a square with sides of length 25 ft.

Exercises:

(i) Find the maximum and minimum values of the following functions:

(a) $f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

(b) $f(x) = 12x^5 - 5x^4 + 40x^3 + 6$

(c) $f(x) = \frac{x^3}{3} + ax^2 - 3a^2x$

(d) $f(x) = 2x^3 - 6x^2 - 18x + 7$

(e) $f(x) = 3x^4 - 20x^3 - 6x^2 + 60x + 15$

(ii) In a Mango orchard, there are 30 Mango trees per acre and each tree yields 400 mangoes. For an excess tree per acre the number of yields reduces by 10. In order to per maximum yield how many trees should be there per acre?

(iii) A farmer can surround his rectangular garden of maximum area by an 800ft long bamboo fence. Find its Length and Width?

(iv) A closed cylindrical can need to be made up with fixed volume. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?

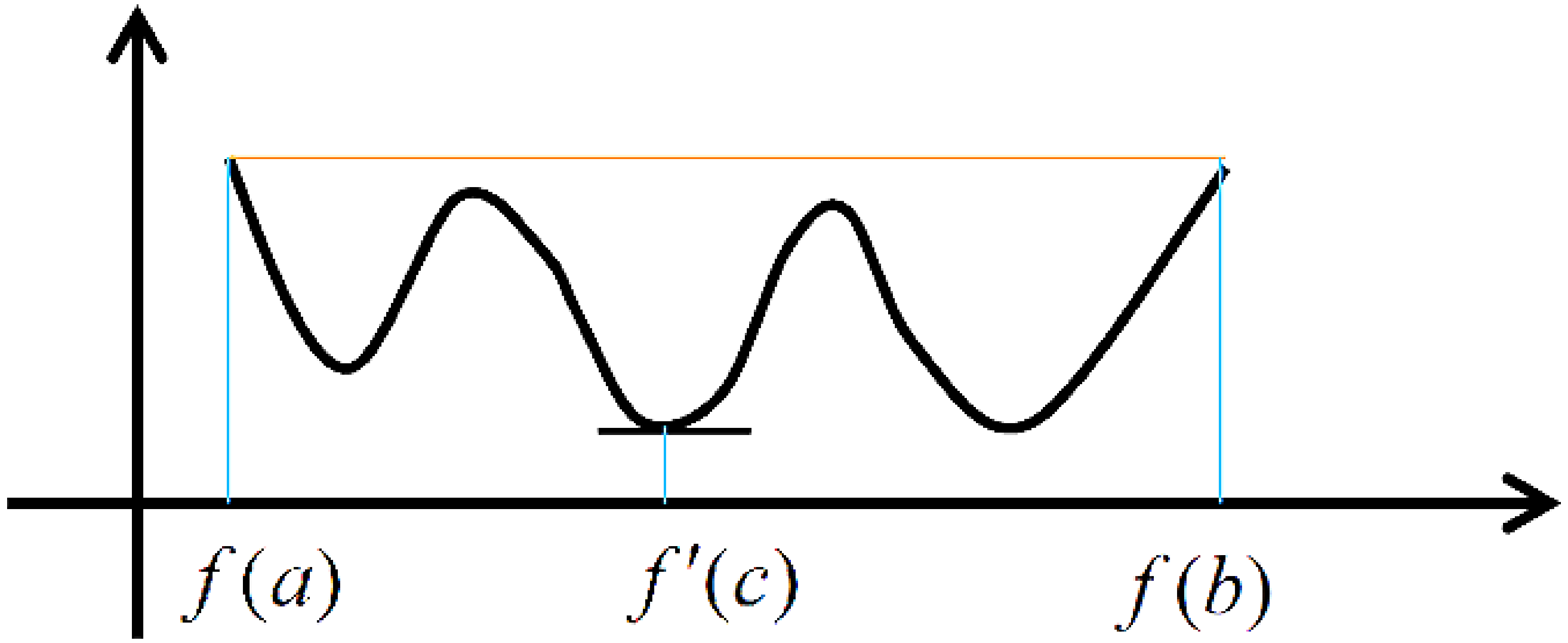
WEAK 11

Page:17-25

Rolle's Theorem:

Let f be continuous on the closed interval $[a,\ b]$ and differentiable on the open interval $(a,\ b)$.

If $f(a) = 0$ and $f(b) = 0$then there is at least one point c in the interval $(a,\ b)$such that $f'(c) = 0$.



Algebraic Significance:

If $f(x)$ be a polynomial in x and $x = a,\ \ x = b$ be two roots of the equation $f(x) = 0$, then from Rolle's theorem we find at least one root of the equation $f'(x) = 0$ lies between 'a' and 'b'.

Geometrical Significance:

From Rolle's Theorem we get a tangent parallel to x-axis at the point 'c' between 'a' and 'b'.

Question 1:

Verify that the function $f(x) = x^2 - 5x + 4$ satisfied the hypotheses of Rolle's Theorem are on the interval [1,4], and find all values of *c* in that interval that satisfy the conclusion of the theorem.

Solution:

Given the function is as follows, $f(x) = x^2 - 5x + 4$

The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[1,4]$ and differentiable on $(1,4)$.

$$\text{Also } f(1) = 1^2 - 5 \times 1 + 4 = 0$$

$$\text{and } f(4) = 4^2 - 5 \times 4 + 4 = 0$$

So, the hypotheses of Rolle's Theorem are satisfied on the interval [1, 4].

Differentiating the given function $f(x)$ with respect to x we get

$$f'(x) = 2x - 5.$$

Let $c \in (1,4)$ so that $f'(c) = 2c - 5.$

From Rolle's Theorem we have $f'(c) = 0$

$$\Rightarrow 2c - 5 = 0$$

$$\therefore c = \frac{5}{2} \in (1,4)$$

So $c = \frac{5}{2}$ is a point in the interval $(1,4)$ at which $f'(c) = 0$

Question 2:

Verify that the function $f(x) = 2x^3 + x^2 - 4x - 2$ satisfied the hypotheses of Rolle's Theorem are on the interval $[-\sqrt{2}, \sqrt{2}]$, and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution:

Given the function is as follows, $f(x) = 2x^3 + x^2 - 4x - 2$

The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[-\sqrt{2}, \sqrt{2}]$ and differentiable on $(-\sqrt{2}, \sqrt{2})$.

$$\text{Also } f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2 = -4\sqrt{2} + 2 + 4\sqrt{2} - 2 = 0$$

$$\text{and } f(\sqrt{2}) = 2(\sqrt{2})^3 + (\sqrt{2})^2 - 4\sqrt{2} - 2 = 4\sqrt{2} + 2 - 4\sqrt{2} - 2 = 0$$

So, the hypotheses of Rolle's Theorem are satisfied on the interval $[-\sqrt{2}, \sqrt{2}]$.

Differentiating the given function $f(x)$ with respect to x we get

$$f'(x) = 6x^2 + 2x - 4$$

Let $c \in (-\sqrt{2}, \sqrt{2})$ so that $f'(c) = 6c^2 + 2c - 4$

From Rolle's Theorem we have $f'(c) = 0$

$$\Rightarrow 6c^2 + 2c - 4 = 0$$

$$\Rightarrow 3c^2 + c - 2 = 0$$

$$\Rightarrow 3c^2 + 3c - 2c - 2 = 0$$

$$\Rightarrow 3c(c+1) - 2(c+1) = 0$$

$$\Rightarrow (c+1)(3c-2) = 0$$

$$\therefore c = -1, \tfrac{2}{3} \in (-\sqrt{2}, \sqrt{2})$$

It shows that we get two c in the interval $(-\sqrt{2}, \sqrt{2})$ at which $f'(c) = 0$.

Question 3:

Verify that the function $f(x) = (x-1)(x-2)(x-3)$ satisfied the hypotheses of Rolle's Theorem are on the interval $[1,3]$, and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution:

Given function is as follows,

$$f(x) = (x-1)(x-2)(x-3)$$

The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[1,3]$ and differentiable on $(1,3)$.

$$\text{so, } f(1) = (1-1)(1-2)(1-3) = 0$$

$$\text{and } \quad f(3) = (3-1)(3-2)(3-3) = 0$$

$$\therefore f(1) = f(3)$$

So, the hypotheses of Rolle's Theorem are satisfied on the interval [1,3].

We have,

$$f(x) = (x^2 - 3x + 2)(x-3)$$

$$\Rightarrow f(x) = x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$$

$\therefore\ f(x) = x^3 - 6x^2 + 11\text{x} - 6$

Differentiating the given function $f(x)$ with respect to x we get,

$$f'(x) = 3x^2 - 12x + 11$$

Let $c \in (1,3)$ so that $f'(c) = 3c^2 - 12c + 11$

From Rolle's Theorem we have, $f'(c) = 0$

$$\Rightarrow 3c^2 - 12c + 11 = 0$$

$$\therefore c = \frac{12 \pm \sqrt{144 - 4.3.11}}{2.3} = \frac{12 \pm \sqrt{12}}{6}$$

Taking positive sign $c = 2.57 \in (1,3)$ and also taking negative sign $c = 1.42 \in (1,3)$

It shows that we get two $c \in (1,3)$ such that $f'(c) = 0$.

Exercise:

Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval and find all values of *c* in that interval that satisfy the conclusion of the theorem.

(i) $f(x) = x^3 - 6x^2 + 11x - 6; [-\sqrt{5}, \sqrt{5}]$

(ii) $f(x) = x^2 + 5x - 6; [-6,1]$

(iii) $f(x) = (x-1)(x-2); [1,2]$

(iv) $f(x) = x^2 - 3x + 2; [1,2]$

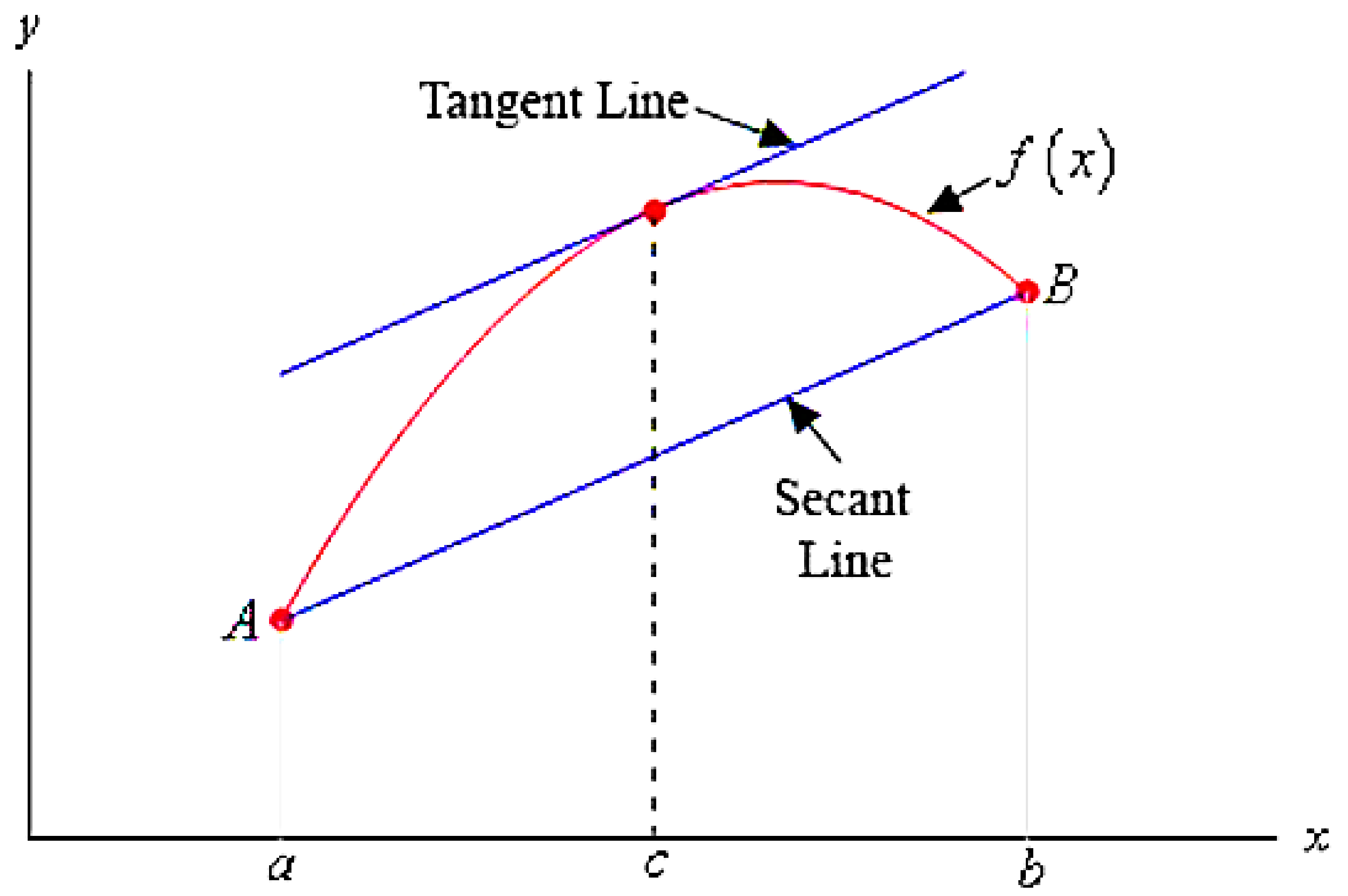
Mean-Value Theorem:

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

Then there is at least one point c in $(a,\ b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Geometrical Significance:

The tangent to the curve at ‘c’ is parallel to the chord joining to two ends points.



Question 1:

Verify that the function $f(x) = x^3 - 6x^2 + 11x - 6$ satisfied hypotheses of the Mean-Value Theorem are on the interval $[0,\ 4]$ and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution:

Given function is as follows,

$$f(x) = x^3 - 6x^2 + 11x - 6$$

The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0,4]$ and differentiable on $(0,\ 4)$. So the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 4$.

$$\text{Now } f(a) = f(0) = 0^3 - 6 \times 0^2 + 11 \times 0 - 6 = -6$$

$$\text{and } f(b) = f(4) = 4^3 - 6 \times 4^2 + 11 \times 4 - 6 = 6$$

$$\text{Here } f'(x) = 3x^2 - 12x + 11. \text{ Therefore } f'(c) = 3c^2 - 12c + 11$$

From Mean Value Theorem, we have $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{6+6}{4-0} = \frac{12}{4} = 3$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144-96}}{6}$$

Thus $C = 0.845,\quad 3.155$ are lying in the interval $(0,\ 4)$

Question 1:

Verify that the function $f(x) = x(x-1)(x-2)$ satisfied hypotheses of the Mean-Value Theorem are on the interval $[0, 1]$ and find all values of c in that interval that satisfy the conclusion of the theorem.

Solution:

Given function is as follows,

$$f(x) = x(x-1)(x-2)$$

The function f is continuous and differentiable everywhere because it is a polynomial. In particular, f is continuous on $[0,1]$ and differentiable on $(0, 1)$. So the hypotheses of the Mean-Value Theorem are satisfied with $a = 0$ and $b = 1$.

$$\therefore f(a) = f(0) = 0.(0-1)(0-2) = 0$$

$$\text{And } f(b) = f(1) = 1(1-1)(1-2) = 0$$

Again, we have

$$f(x) = x(x-1)(x-2)$$

$$f(x) = (x^2 - x)(x-2)$$

$$\Rightarrow f(x) = x^3 - x^2 - 2x^2 - 2x$$

$$\Rightarrow \ f(x) = x^3 - 3x^2 - 2x$$

Differentiating the given function f(x) with respect to x we get,

$$f'(x) = 3x^2 - 6x + 2$$

Let $c \in (1,3)$ so that $\quad f'(c) = 3c^2 - 6c + 2$

From Mean value Theorem we have,

$$f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{0-0}{1-0} = 0$$

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 6c + 2 = 0$$

$$\Rightarrow c = \frac{6 \pm \sqrt{36-4.3.2}}{2.3} = \frac{6 \pm \sqrt{12}}{6}$$

Taking positive sign $c = 1.57 \notin (0.1)$ and also taking negative sign $c = 0.42 \in \ (0,1)$

It shows that we get one $c \in (0,1)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Exercise:

Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(i) $f(x) = x^3 - 4x$; $[-2,1]$

(ii) $f(x) = x^3 + x - 2$; $[-1,2]$

(iii) $f(x) = x^3 + 3x^2 - 5x$; $[1,2]$

(iii) $f(x) = x(x-1)(x-2)$; $[0, \frac{1}{2}]$

WEAK 12

Page:17-25

Concept of Integration

A function F is called an ***antiderivative*** of a function f on a given open interval if $\boldsymbol{F'(x) = f(x)}$ for all x in the interval. The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if $\frac{d}{dx}[F(x)] = f(x) \cdots\cdots \quad (1)$ then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***, $\int f(x)dx = F(x) + C \quad \cdots\cdots \quad (2)$ where C is understood to represent an **arbitrary constant**. It is important to note that (1) and (2) are just different notations to express the same fact.

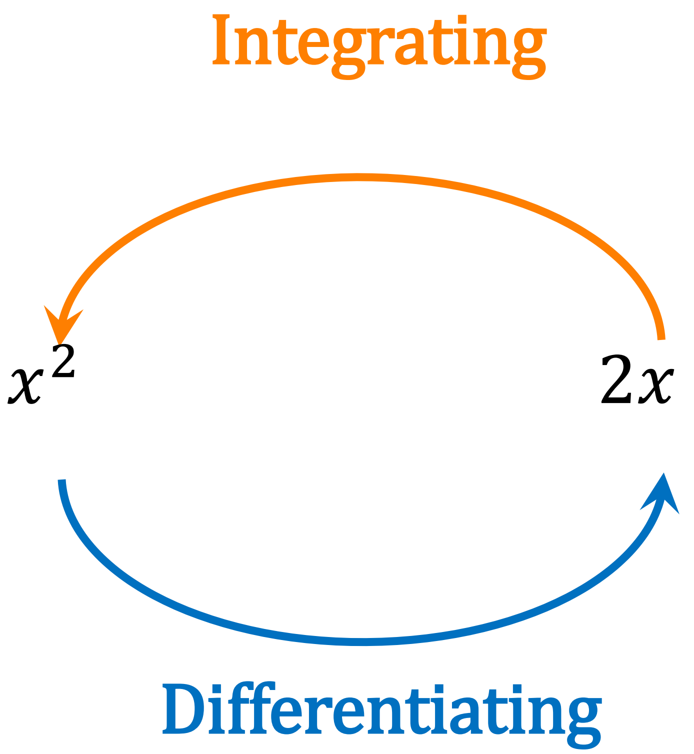
For example, $\int x^2 dx = \frac{1}{3}x^3 + C$ is equivalent to $\frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = x^2$

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\int f(x)\, dx = F(x)$$

$$\frac{d}{dx}[x^2] = 2x$$

$$\int 2x\, dx = x^2$$



$$\frac{d}{dx}\left[\int f(x)\, dx\right] = f(x)$$

$$\frac{d}{dx}\left[\int (2x)\, dx\right] = \frac{d}{dx}[x^2] = 2x$$

$$\int f(x)\,dx = F(x) + C$$

Constant of integration

Integral Symbol

Integrating with respect to x

$$\int f(x)\,dx$$

Integrand

(function we want to integrate)

Why "C" ???

Integrating

$x^2 + 4, x^2,$
$x^2 + 100, \quad x^2 - 6$

$2x$

Differentiating

$$\int 2x\,dx = x^2 + C$$

Some Important Formulae:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
2. $\int 1 dx = x + c$
3. $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$
4. $\int e^{mx} dx = \frac{e^{mx}}{m} + c$
5. $\int a^x dx = \frac{a^x}{\ln a} + c$
6. $\int \frac{1}{x} dx = \ln x + c$
7. $\int \sin x\, dx = -\cos x + c$
8. $\int \cos x\, dx = \sin x + c$
9. $\int \sec^2 x\, dx = \tan x + c$
10. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
11. $\int \sec x \tan x\, dx = \sec x + c$
12. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
13. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$
14. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$
15. $\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{cosec}^{-1} x + c$
16. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + +c$
17. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$
18. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$
19. $\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$
20. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$
21. $\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$
22. $\int [f(x)]^n . f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c$

Rule 1:

$$\int cf(x)\,dx = c\int f(x)\,dx$$

Example:

Evaluate

(i) $\int 3\cos x\, dx$

(ii) $\int \frac{3}{\sqrt{x}}\, dx$

Solution:

(i) $\int 3\cos x\, dx$

$= 3\int \cos x\, dx$

$\because \int \cos x\, dx = \sin x + c$

$= 3\sin x + c$

(ii) $\int \frac{3}{\sqrt{x}}\, dx$

$= 3\int x^{-\frac{1}{2}}\, dx$

$= \frac{3x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$

$\because \int x^n dx = \frac{x^{n+1}}{n+1} + c$

$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 6x^{\frac{1}{2}} + c$

Rule 2:

$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

Example 1:

Evaluate $\int \left(2\sin 3x + 4\cos\frac{9x}{2} + 5x^3 + 3e^{-\frac{4}{5}x} \right) dx$

Solution:

$$\int \left[2\sin 3x + 4\cos\frac{9}{2}x + 5x^3 + 3e^{-\frac{4}{5}x} \right] dx$$

$$= \int \sin 3x\ dx + 4\int \cos\frac{9}{2}x\ dx + 5\int x^3 dx + 3\int e^{-\frac{4}{5}x}\, dx$$

$$= \frac{2}{-3}\cos 3x + \frac{4}{\frac{9}{2}}\sin\frac{9}{2}x + \frac{5}{4}x^4 + \frac{3}{-\frac{4}{5}}e^{-\frac{4}{5}x} + c$$

$$= -\frac{2}{3}\cos 3x + \frac{8}{9}\sin\frac{9}{2}x + \frac{5}{4}x^4 - \frac{15}{4}e^{-\frac{4}{5}x} + c$$

$\int \cos mx\, dx = \frac{\sin mx}{m} + c$

$\int \sin mx\, dx = \frac{-\cos mx}{m} + c$

$\int e^{mx}\, dx = \frac{e^{mx}}{m} + c$

Example 2:

Evaluate (i) $\int (e^x - 5a^x + 2)\, dx$ (ii) $\int \frac{x+2\sqrt{x}+7}{\sqrt{x}}$

Solution:

(i) $\int (e^x - 5a^x + 2)\, dx$

$= \int e^x\, dx - \int 5a^x\, dx + \int 2\, dx$

$= e^x - 5\int a^x\, dx + 2\int dx$

$= e^x - 5\frac{a^x}{\ln a} + 2x + C$

(ii) $\int \frac{x+2\sqrt{x}+7}{\sqrt{x}}\, dx$

$= \int \left(\frac{x}{\sqrt{x}} + \frac{2\sqrt{x}}{\sqrt{x}} + \frac{7}{\sqrt{x}}\right) dx$

$= \int \left(x^{\frac{1}{2}} + 2 + 7x^{-\frac{1}{2}} \right) dx$

$= \int x^{\frac{1}{2}}\, dx + \int 2\, dx + \int 7x^{-\frac{1}{2}}\, dx$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 2x + 7\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2x + 7\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{3}(\sqrt{x})^3 + 2x + 14\sqrt{x} + C$

$\int e^x\, dx = e^x + c$

$\int a^x\, dx = \frac{a^x}{\ln a} + c$

$\int dx = x + c$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

Type:

Integration

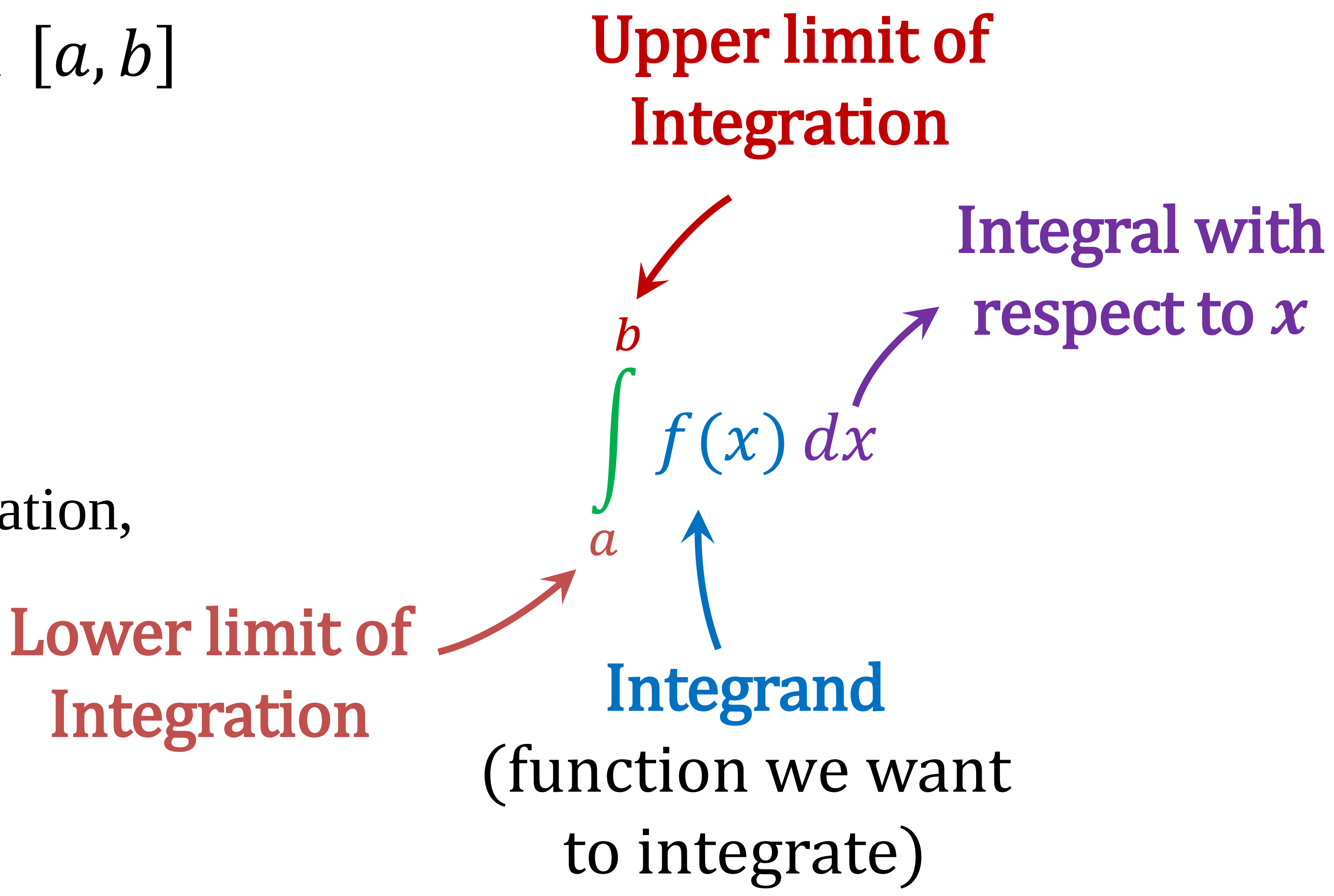
- Definite Integration: $$\int_{-1}^{3} x^2\,dx$$
- Indefinite Integration: $$\int x^2\,dx$$

Definite Integral

If $f(x)$ is a continuous function defined in the interval $[a, b]$ then the definite integral with respect to x is defined as,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Where a and b are called lower and upper limits of integration, respectively



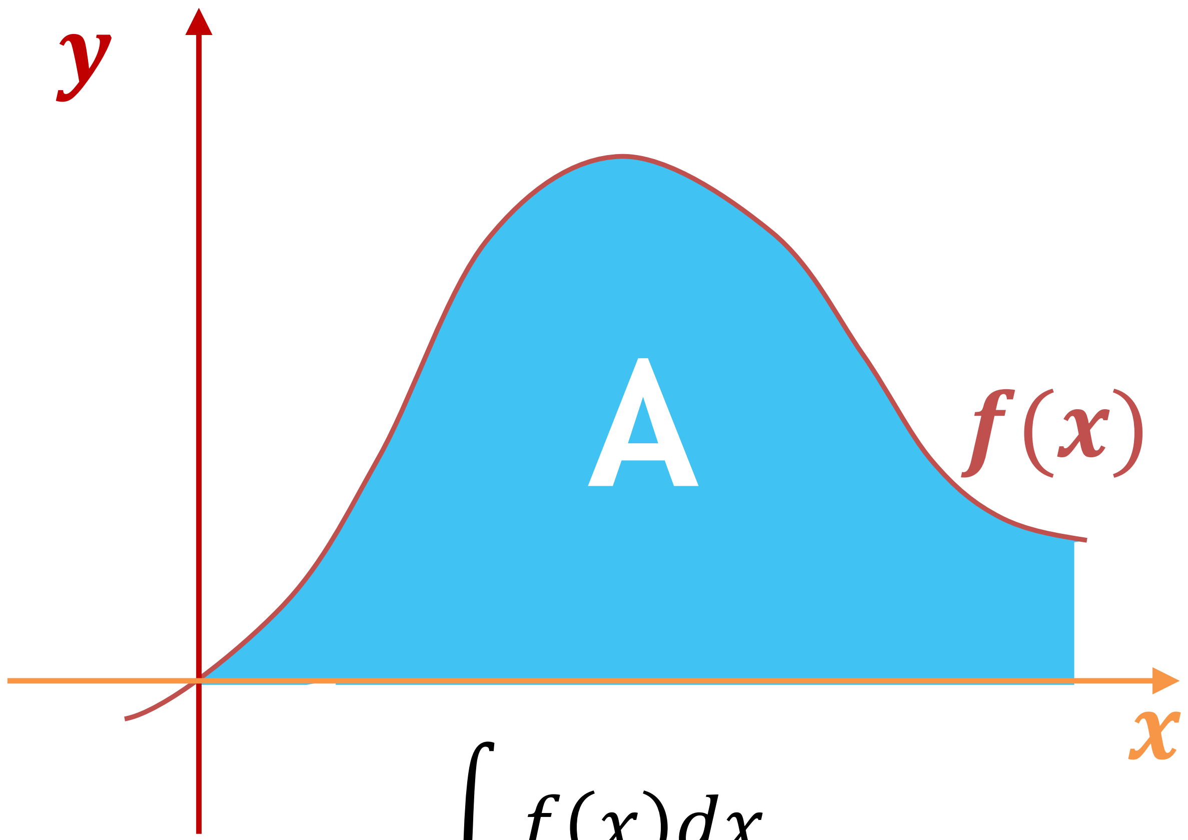
This formula is known as Newton-Leibnitz formula

Note:

- The indefinite integral $\int f(x)\,dx$ is a function of x, whereas definite integral $\int_a^b f(x)\,dx$ is a number.
- Given $\int f(x)\,dx$ we can find $\int_a^b f(x)\,dx$ but given $\int_a^b f(x)\,dx$ we cannot find $\int f(x)\,dx$.

Geometrical Interpretation

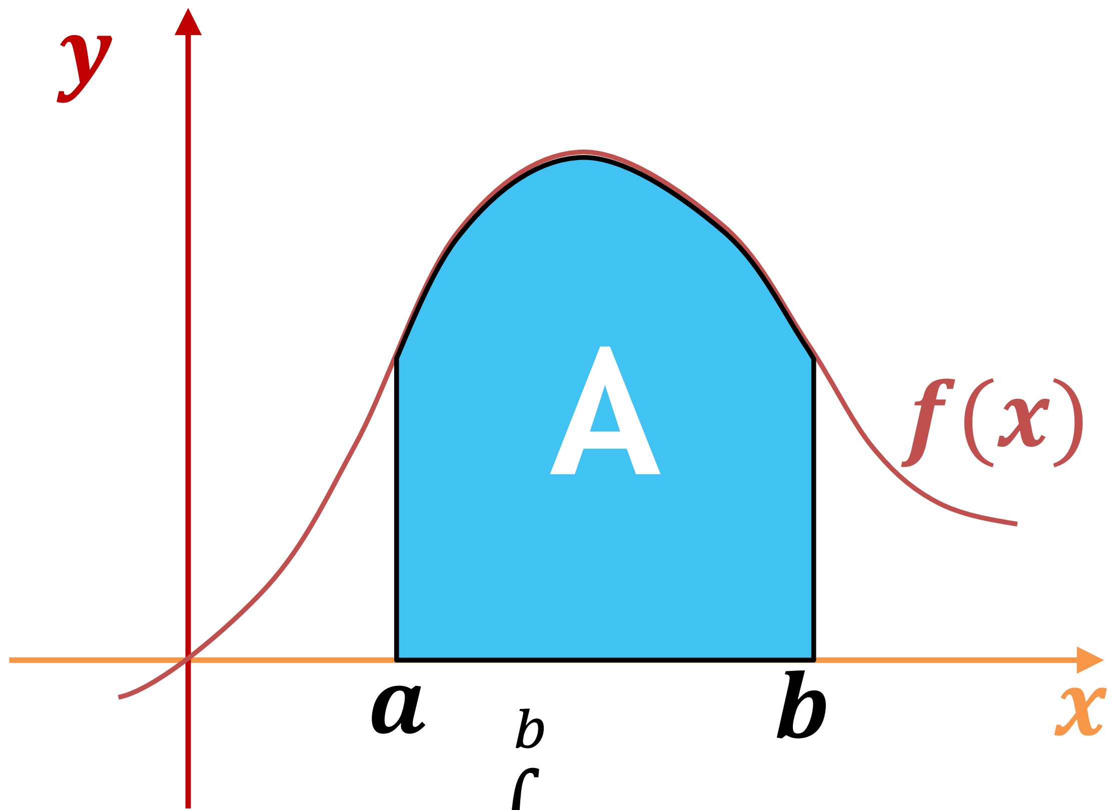
Integration can be used to find areas, volumes, central points and many useful things. But it is often used to find the **area under the graph of a function** like this:



$$\int f(x)dx$$

Indefinite Integral

(no specific values)



$$\int_a^b f(x)dx$$

Definite Integral

(from a to b)

Definite integral has start and end values: in other words there is an interval $[a, b]$. We can find out the actual area under a curve

Properties of Definite integral

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2) $\int_a^b f(x)dx = \int_a^b f(z)dz$

3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ $when\ a < e < b.$

Example:

Evaluate $(i)\ \int_{-3}^{1}(6x^2 - 5x + 2)dx$ $(ii)\ \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

Solution:

$(i)\ \int_{-3}^{1}(6x^2 - 5x + 2)dx$

$$= \left[6\frac{x^3}{3} + 5\frac{x^2}{2} - 2x\right]_{-3}^{1}$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$= \left(2\times1^3+5\frac{1^2}{2}-2\times1\right)-\left(2\times(-3)^3+5\frac{(-3)^2}{2}-2\times(-3)\right)$$

$$= \left(2+\frac{5}{2}-2\right)-\left(-54+\frac{45}{2}+6\right)$$

$$= \frac{5}{2}-\frac{51}{2}=\frac{5}{2}-\frac{51}{2}=-\frac{46}{2}$$

$$= -23$$

(ii) $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\, dx$$

$$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+\frac{\sin 2\frac{\pi}{2}}{2}\right)-\left(0+\frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]=\frac{\pi}{4}$$

$$\int \cos mx\, dx = \frac{\sin mx}{m}+c$$

$$\int dx = x+c$$

WEAK 13

Page:17-25

Method of Substitution:

Integration by substitution, also known as "u-substitution", is a technique that simplifies integrals by making a substitution. The exact substitution depends on the integral's form, but here are some steps you can follow:

1. Identify the integral in the form $\int f(g(x))g'(x)dx$
2. Substitute the independent variable as $g(x) = t$
3. Differentiate the assumed function with respect to t
4. Substitute for the dependent variable, such as $g'(x)dx = dt$
5. The resultant integral after substitution becomes $\int f(t)dt$
6. Solve the integral using basic integration rules
7. Convert the result back to terms of x by substituting t with the original independent variable

If integrand $f(x) = g(x).g'(x)$ or $[g(x)]^n g'(x)$ or $e^{g(x)}.g'(x)$, $T[g(x)].g'(x)$ or $[T(g(x))]^n.g'(x)$ then substitute $g(x) = z$ and $g'(x)dx = dz$. Here T means trigonometry function.

Example:

Evaluate the following indefinite integral

(i) $\int \frac{e^x(x+1)}{cos^2(x\,e^x)}dx$ (ii) $\int x^9\, e^{10}\, dx$ (iii) $\int_0^1 x^3\sqrt{1+3x^4}\ dx$ (iv) $\int_0^{\frac{\pi}{4}}(tan^5 x + tan^3 x)\, dx$

Solution:

(i) $\int \frac{e^x(x+1)}{cos^2(x\,e^x)}dx$

$= \int \frac{dz}{cos^2 z}$

$= \int sec^2 z\, dz$

$= tan\, z + c$

$= tan\,(x\, e^x) + c$

Let, $x.e^x = z$

Then $(x.e^x + e^x.1)dx = dz$

$\Rightarrow e^x(x+1)dx = dz$

(ii) $\int x^9\, e^{10}\, dx$

$= \int e^z \frac{dz}{10}$

$= \frac{1}{10}\int e^z\, dz$

$= \frac{1}{10}\,.e^z + c$

$= \frac{1}{10}\, e^{x^{10}} + c$

Let, $x^{10} = z$

$\Rightarrow 10x^9dx = dz$

$\Rightarrow x^9dx = \frac{dz}{10}$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}[x^n] = nx^{n-1}$

$\int e^x\, dx = e^x + c$

$\int \sec^2 x\, dx = \tan x + c$

(iii) $\int_0^1 x^3\sqrt{1+3x^4}\ dx$

$$= \int_1^4 \sqrt{z}\ \frac{dz}{12}$$

$$= \frac{1}{12}\int_1^4 z^{\frac{1}{2}}\, dz$$

$$= \frac{1}{12}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4$$

$$= \frac{1}{12}\left[\frac{z^{3/2}}{\frac{3}{2}}\right]_1^4$$

$$= \frac{1}{12}\times\frac{2}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

$$= \frac{1}{18}[8-1] = \frac{7}{18}$$

Let, $1 + 3x^4 = z$

$\Rightarrow (0 + 3.4x^3)dx = dz$

$\Rightarrow x^3 dx = \frac{dz}{12}$

Limits

when $x = 0$, then $z = 0$

when $x = 1$, then $z = 4$

(iv) $\int_0^{\frac{\pi}{4}} (tan^5 x + tan^3 x)\, dx$

$$= \int_0^{\frac{\pi}{4}} tan^3 x\,(1 + tan^2 x)\ dx$$

$$= \int_0^{\frac{\pi}{4}} tan^3 x\, sec^2 x\ dx$$

$$= \int_0^1 u^3 du$$

$$= \left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{4}[1^4 - 0]$$

$$= \frac{1}{4}$$

Let, $u = tan\, x$

$\therefore\ du = sec^2 x\ dx$

Limits

when $x = 0$, then $u = 0$

when $x = \frac{\pi}{4}$, then $u = 1$

$\sec^2 x - \tan^2 x = 1$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

Some Ideal Integrals

$$\int \frac{f'(x)}{f(x)} dx = ln|f(x)| + c$$

Example:

Evaluate the following indefinite integral **(i)** $\int \frac{tanx}{\ln(cosx)} dx$ **(ii)** $\int_0^1 \frac{x}{4-x^2} dx$

Solution:

(i) $\int \frac{tanx}{\ln(cosx)} dx$

$= -\int \frac{-tanx}{\ln(cosx)} dx$

$= -\ln(lncosx) + c$

Here $f(x) = \ln(cosx)$

And $f'(x) = \frac{1}{c0sx}.(-sinx)$

$= -tanx$

(ii) $\int_0^1 \frac{x}{4-x^2} dx$

$= \left(-\frac{1}{2}\right) \int_0^1 \frac{-2x}{4-x^2} dx$

$= -\frac{1}{2}[ln\,|4 - x^2|]_0^1$

$= -\frac{1}{2}\{\ln(4 - 1^2) - \ln(4 - 0)\}$

$= -\frac{1}{2}\{ln3 - ln4\} = -\frac{1}{2}\ln(\frac{3}{4})$

Here $f(x) = 4 - x^2$

And $f'(x) = -2x$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$$

Example:

Evaluate the following indefinite integral **(i)** $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$ **(ii)** $\int \frac{\sec^2 x}{\sqrt{1+tanx}}dx$

Solution:

(i) $\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx$

$= \left(-\frac{1}{2}\right)\int_0^1 \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= -\frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^1$

$= -\{\sqrt{1-1} - \sqrt{1-0}\}$

$= 1$

Here $f(x) = 1 - x^2$

And $f'(x) = -2x$

(ii) $\int \frac{\sec^2 x}{\sqrt{1+tanx}}dx$

$= 2\sqrt{1+tanx} + c$

Here $f(x) = 1 - x^2$

And $f'(x) = -2x$

$$\int [f(x)]^n . f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

Example:

Evaluate the following indefinite integral **(i)** $\int (cot^7 x + cot^5 x)dx$ **(ii)** $\int_0^{\frac{\pi}{2}} sin^7 x \, cos \, x \; dx$

Solution:

(i) $\int (cot^7 x + cot^5 x)dx$

$=\int cot^5 x \,(cot^2 x + 1)\, dx$

$=\int cot^5 x \, cosec^2 x \, dx$

$= (-)\int cot^5 x \,(-cosec^2 x)\, dx$

$= -\frac{(cotx)^{5+1}}{5+1} + c$

$= -\frac{(cotx)^6}{6} + c$

Here $f(x) = cotx$

And $f'(x) = -\, cosec^2 x$

(ii) $\int_0^{\frac{\pi}{2}} sin^7 x \, cos \, x \; dx$

$= \left[\frac{(sin\, x)^{7+1}}{7+1}\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{1}{8}\, sin^8 x\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{8}\{\sin^8 \frac{\pi}{2} - \sin^8 0\}$

$= \frac{1}{8}$

Here $f(x) = sinx$

And $f'(x) = cosx$

- $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$
- $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + +c$
- $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$
- $\int \sqrt{a^2 - x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Example:

Evaluate the following indefinite integral **(i)** $\int \frac{dx}{16-4x^2}$ **(ii)** $\int_0^1 \frac{dx}{\sqrt{2x-x^2}}$

Solution:

(i) $\int \frac{dx}{16-4x^2}$

$= \int \frac{dx}{4(4-x^2)}$

$= \frac{1}{4}\int \frac{dx}{2^2-x^2}$

$= \frac{1}{4}.\frac{1}{2\times 2}\ln\left|\frac{2+x}{2-x}\right| + c$

$= \frac{1}{16}\ln\left|\frac{2+x}{2-x}\right| + c$

(ii) $\int_0^1 \frac{dx}{\sqrt{2x-x^2}}$

$= \int_0^1 \frac{dx}{\sqrt{1-(1-2x+x^2)}}$

$= \int_0^1 \frac{dx}{\sqrt{1-(1-x)^2}}$

$= [\sin^{-1}(x-1)]_0^1$

$= \{\sin^{-1}(1-1) - \sin^{-1}(0-1)\}$

$= 0 + \frac{\pi}{2}$

$= \frac{\pi}{2}$

Type: $\int \frac{dx}{a\cos^2x+b\sin^2x+c}$; $\int \frac{dx}{a\cos^2x+c}$; $\int \frac{dx}{a\cos^2x+b\sin^2x}$

In this case, we will bring $\tan^2x$ in the denominator. So, the $\sin^2x$ in denominator will be divided by $\cos^2x$. So the numerator and denominator will be divided by $\cos^2x$ and then in the numerator, dx will remain in product with $\sec^2x$ and in the denominator there will be $\tan^2x$.Now, we will take $tanx = z$ and integrate.

Example: $\int \frac{dx}{4\cos^2x + 9\sin^2x}$

Solution:

$$\int \frac{dx}{4\cos^2x+9\sin^2x}$$

$$= \int \frac{\frac{1}{\cos^2x}dx}{\frac{4\cos^2x}{\cos^2x}+\frac{9\sin^2x}{\sin^2x}}$$

$$= \int \frac{\sec^2x\,dx}{4+9\tan^2x}$$

$$= \int \frac{dz}{4+9z^2}$$

Let $tanx = z$

$\therefore \sec^2 x\, dx = dz$

$$= \int \frac{dz}{4+9z^2}$$

$$= \int \frac{dz}{9(\frac{4}{9}+z^2)}$$

$$= \frac{1}{9}\int \frac{dz}{(\frac{2}{3})^2+z^2}$$

$$= \frac{1}{9} \times \frac{3}{2}\tan^{-1}\frac{z}{\frac{2}{3}}$$

$$= \frac{1}{6}\tan^{-1}\frac{3z}{2} + c$$

$$= \frac{1}{6}\tan^{-1}(\frac{3}{2}\tanx) + c$$

Type: $\int \frac{(px+q)}{ax^2+bx+c}\,dx$, $\int \frac{(px+q)}{\sqrt{ax^2+bx+c}}\,dx$, $\int (px+q)\sqrt{ax^2+bx+c}\;dx$

For this kind of integral, we need to follow the following steps sequentially.

Expressing in the form of $\lambda \frac{d}{dx}(ax^2+bx+c)+s = px+q$ → Comparing the coefficent determine the value of λ & s → Then, Integrate

Example:

$$\int \frac{x+5}{x^2+4x+13}\,dx$$

Solution:

Let $x+5 = \lambda \frac{d}{dx}(x^2+4x+13)+s$

$\Rightarrow\; x+5 = \lambda(2x+4)+s$

Equating the coefficients of x and constant we get,

$1 = 2\lambda$ and $5 = \lambda \times 4 + s$

$\Rightarrow \lambda = \frac{1}{2}$ $\Rightarrow 5 = \frac{1}{2} \times 4 + s \;\Rightarrow s = 3$

$$\therefore \int \frac{x+5}{x^2+4x+13} dx = \int \frac{\frac{1}{2}(2x+4)+3}{x^2+4x+13} dx$$

$$= \frac{1}{2} \int \frac{(2x+4)}{x^2+4x+13} dx + 3 \int \frac{dx}{x^2+4x+13}$$

$$= \frac{1}{2} \ln|x^2 + 4x + 13| + 3.\frac{1}{3} tan^{-1} \left(\frac{x+2}{3}\right) + c$$

$$= \frac{1}{2} \ln|x^2 + 4x + 13| + tan^{-1} \left(\frac{x+2}{3}\right) + c$$

Type: In the form of $a^2 \pm x^2$ & $x^2 \pm a^2$

Concept: (i) If $f(x) = a^2 + x^2$, then, we will take $x = a\tan\theta$

(ii) If $f(x) = a^2 - x^2$, then, we will take $x = a\sin\theta$

(iii) If $f(x) = x^2 - a^2$, then, we will take $x = a\sec\theta$

Example:

$$\int \frac{dx}{(x^2+9)^2}$$

Solution:

$$\int \frac{dx}{(x^2+9)^2}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{(3\tan^2\theta+3^2)^2}$$

$$= \int \frac{3\sec^2\theta\, d\theta}{9^2(1+\tan^2\theta)}$$

Suppose, $x=3\tan\theta$

$\Rightarrow dx = 3\sec^2\theta d\theta$

$$= \frac{1}{27}\int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2}$$

$$= \frac{1}{27}\int \frac{d\theta}{\sec^2\theta}$$

$$= \frac{1}{27\times 2}\int 2.\cos^2\theta \, d\theta$$

$$= \frac{1}{54}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{54}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{1}{54}\left[\theta + \frac{1}{2}.\frac{2\tan\theta}{1+\tan^2\theta}\right] + c$$

$$= \frac{1}{54}\left[\theta + \frac{\tan\theta}{1+\tan^2\theta}\right] + c$$

$$= \frac{1}{54}\left[\tan^{-1}\left(\frac{x}{3}\right) + \frac{\frac{x}{3}}{1+\frac{x^2}{9}}\right] + c$$

$$= \frac{1}{54}\left[\tan^{-1}\left(\frac{x}{3}\right) + \frac{\frac{x}{3}}{\frac{9+x^2}{9}}\right] + c$$

$$= \frac{1}{54}\left[\tan^{-1}\left(\frac{x}{3}\right) + \frac{3x}{9+x^2}\right] + c$$

WEAK 14

Page:17-25

Integration by Parts:

If u and v are function of x then

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

Guide linxe for selecting u xand v:x

L	I	A	T	E

L : Logoritmic function $(\log x, \ln x, \ldots)$

I : Inverse Trigonometric Function$(\sin^{-1} x, \tan^{-1} x, \ldots)$

A : Algebraic Function $(2x, x^2 + 7x, x^{-5}, \ldots)$

T : Trigonometric Function$(\cos x, \sec x, \ldots)$

E : Exponential Function$(e^x, e^{-2x}, \ldots)$

Choose “u” to be the function that comes first in this list

Example:

Evaluate the following indefinite integral

(i) $\int xe^x\,dx$ (ii) $\int x\ln x\,dx$ (iii) $\int \ln x\,dx$

Solution:

(i) $\int xe^x\,d$

$$= x\int e^x dx - \int\left\{\frac{d}{dx}(x)\int e^x dx\right\}dx$$

$$= xe^x - \int 1.e^x dx$$

$$= xe^x - e^x + c$$

$$\therefore \int xe^x\,d = xe^x - e^x + c$$

$$\int \boldsymbol{uv\,dx} = \boldsymbol{u}\int \boldsymbol{v\,dx} - \int\{\frac{\boldsymbol{du}}{\boldsymbol{dx}}\int \boldsymbol{v\,dx}\}\boldsymbol{dx}$$

L I A T E

L : Logoritmic function
I : Inverse Trigonometric Function
A : Algebraic Function
T : Trigonometric Function
E : Exponential Function

(ii) $\int x ln x \, dx$

$$= ln x \int x dx - \int \left\{ \frac{d}{dx}(ln x) \int x dx \right\} dx$$

$$= \frac{x^2}{2} ln x - \int \frac{1}{x} . \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} ln x - \frac{1}{2} \int x dx$$

$$= \frac{x^2}{2} ln x - \frac{x^2}{4} + c$$

$$\therefore \int x ln x \, dx = \frac{x^2}{2} ln x - \frac{x^2}{4} + c$$

$$\int \boldsymbol{uv \, dx = u \int v \, dx - \int \{\frac{du}{dx} \int v \, dx\} dx}$$

L I A T E

L : Logoritmic function

I : Inverse Trigonometric Function

A : Algebraic Function

T : Trigonometric Function

E : Exponential Function

(iii) $\int lnx\, dx$

$$= \int lnx\,.1\, dx$$

$$= lnx \int 1dx - \int \left\{\frac{d}{dx}(lnx) \int 1dx\right\} dx$$

$$= xlnx - \int \frac{1}{x}.xdx$$

$$= xlnx - \int 1dx$$

$$= xlnx - x + c$$

$$\therefore \int lnx\, dx = xlnx - x + c$$

$$\int \boldsymbol{uv}\, \boldsymbol{dx} = \boldsymbol{u} \int \boldsymbol{v}\, \boldsymbol{dx} - \int \{\frac{\boldsymbol{du}}{\boldsymbol{dx}} \int \boldsymbol{v}\, \boldsymbol{dx}\}\boldsymbol{dx}$$

L I A T E

L : Logoritmic function
I : Inverse Trigonometric Function
A : Algebraic Function
T : Trigonometric Function
E : Exponential Function

Special Formula (i) $$\int e^{ax}[af(x) + f'(x)]\, dx = e^{ax} f(x) + c$$

(ii) $$\int e^{x}[f(x) + f'(x)]\, dx = e^{x} f(x) + c$$

Example:

Evaluate the following indefinite integral

(i) $\int e^{5x}\left\{5lnx + \frac{1}{x}\right\} dx$ (ii) $\int \frac{e^x(x^2+1)}{(x+1)^2} dx$ (iii) $\int e^x\left(\frac{1+sinx}{1+cosx}\right)dx$

Solution:

(i) $\int e^{5x}\left\{5lnx + \frac{1}{x}\right\} dx$

$= e^{5x} lnx + c$

Here $f(x) = lnx$

And $f'(x) = \frac{1}{x}$

$a = 5$

(ii) $\int \frac{e^x(x^2+1)}{(x+1)^2}dx$

$= \int \frac{e^x\{(x^2-1)+2\}}{(x+1)^2}dx$

$= \int e^x\left\{\frac{(x-1)}{(x+1)} + \frac{2}{(x+1)^2}\right\}dx$

$= \int e^x[f(x) + f'(x)]dx$

$= e^x f(x) + c$

$= e^x.\left(\frac{x-1}{x+1}\right) + c$

Let, $f(x) = \frac{x-1}{x+1}$

$\therefore f'(x) = \frac{(x+1).1-(x-1).1}{(x+1)^2}$

$= \frac{2}{(x+1)^2}$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

(iii) $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$= \int e^x\left(\left(\frac{1+2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)\right)dx$

$= \int e^x\left(\frac{1}{2\cos^2\frac{x}{2}} + \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}.\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}\right)dx$

$= \int e^x\left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right)dx$

$= \int e^x[f(x) + f'(x)]dx$

$= e^x f(x) + c$

$= e^x \tan\frac{x}{2} + c$

Let, $f(x) = \tan\frac{x}{2}$

$\therefore f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$

Definite Integral's Special Properties:

1) $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

2) $\int_0^{na} f(x)dx = n\int_0^a f(x)dx$, when $f(a+x) = f(x)$.

3) $\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & if\ f(x)\ is\ even\ function \\ 0 & if\ f(x)\ is\ odd\ function \end{cases}$

Example 1: Evaluate $\int_{-2}^{2} x^9 \left(1 - x^2\right)^7 \, dx$

Solution:

Let I = $\int_{-2}^{2} x^9 (1 - x^2)^7 \, dx$

$\Rightarrow$ I = $\int_{-2}^{2} f(x)\, dx$ Where $f(x) = x^9 (1 - x^2)^7$

Now, $f(-x) = (-x)^9 \{1 - (-x)^2\}^7$

$= -x^9 (1 - x^2)^7$

$= -f(x)$

$\therefore f(x) = x^9 (1 - x^2)^7$ is an odd function

I = $\int_{-2}^{2} x^9 (1 - x^2)^7 \, dx = 0$

$$\int_{-a}^{a} (odd\ function)dx = 0$$

Example 3: **Determine the value of** $\int_0^{\frac{\pi}{2}} \frac{(\tan x)^{2025}}{(\tan x)^{2025}+(\cot x)^{2025}} dx$

Solution:

Let, $I = \int_0^{\frac{\pi}{2}} \frac{(tanx)^{2025}}{(cotx)^{2025}+(tanx)^{2025}} dx \ldots \ldots . (i)$

$$= \int_0^{\frac{\pi}{2}} \frac{\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}^{2025}}{\left\{\cot\left(\frac{\pi}{2}-x\right)\right\}^{2025}+\left\{\tan\left(\frac{\pi}{2}-x\right)\right\}^{2025}} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\cot x)^{2025}}{(tanx)^{2025}+(cotx)^{2025}} dx \ldots \ldots \ldots (ii)$$

Now $(i) + (ii)$ we get

$$I + I = \int_0^{\frac{\pi}{2}} \frac{(tanx)^{2025}}{(\cot x)^{2025} + (tanx)^{2025}} dx + \int_0^{\frac{\pi}{2}} \frac{(\cot x)^{2025}}{(tanx)^{2025}+(cotx)^{2025}} dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} \left[\int_0^{\frac{\pi}{2}} \frac{(tanx)^{2025}}{(\cot x)^{2025}+(tanx)^{2025}} + \frac{(\cot x)^{2025}}{(tanx)^{2025}+(cotx)^{2025}} \right] dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{(tanx)^{2025}+(\cot x)^{2025}}{(\cot x)^{2025}+(tanx)^{2025}} \right] dx$$

$$\Rightarrow 2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$\Rightarrow 2I = [x]_0^{\frac{\pi}{2}}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{(tanx)^{2024}}{(tanx)^{2024}+(cotx)^{2024}} dx = \frac{\pi}{4}$$

Example 3: Find the value of $\int_0^{\pi} \frac{x \sin x}{1+cos^2 x} dx.$

Solution: Let, $I = \int_0^{\pi} \frac{x \sin x}{1+cos^2 x} dx \ldots\ldots\ldots (i)$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+cos^2(\pi-x)} dx$$

$$= \int_0^{\pi} \frac{(\pi-x)\sin(x)}{1+cos^2(x)} dx \ldots\ldots\ldots\ldots (ii)$$

Now $(i) + (ii) \Longrightarrow 2I = \int_0^{\pi} \frac{\pi \sin(x)}{1+cos^2(x)} dx$

$$= -\int_1^{-1} \frac{\pi dz}{z^2+1}$$

$$= \pi \int_{-1}^{1} \frac{dz}{z^2+1}$$

$$= \pi[\tan^{-1} z]_{-1}^{\ 1}$$

Let, $\cos x = z$

$\therefore \sin x dx = -dz$

Limit:

if $x = 0$ then $z = 1$

and if $x = \pi$ then $z = -1$

$$= \pi[\tan^{-1}(1) - \tan^{-1}(-1)]$$
$$= \pi(\frac{\pi}{4} + \frac{\pi}{4})$$
$$\Rightarrow 2I = \frac{\pi^2}{2}$$
$$\Rightarrow I = \frac{\pi^2}{2}$$
$$\therefore \int_0^{\pi} \frac{x \sin x}{1+cos^2 x} dx . = \frac{\pi^2}{4}$$

Example 4:

Find the value of $\int_0^1 \frac{\ln(1+x)}{1+ x^2} dx.$

Solution:

Let I=$\int_0^1 \frac{\ln(1+x)}{1+ x^2} dx$

$$= \int_0^{\frac{\pi}{4}} \frac{\ln(1+tan\theta).\sec^2 \theta \, d\theta}{1+tan^2\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\ln(1+tan\theta).\sec^2 \theta \, d\theta}{sec^2\theta}$$

Let $x = tan\theta$

$\therefore dx = sec^2 \theta d\theta$

Limits:

If $x = 0$ then $0 = tan\theta \Rightarrow \theta = 0$

If $x = 1$ then $1 = tan\theta \Rightarrow \theta = \frac{\pi}{4}$.

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln(1 + tan\theta)\, d\theta \ \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left\{1 + \tan\left(\frac{\pi}{4} - \theta\right)\right\} d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left\{1 + \frac{\tan\left(\frac{\pi}{4}\right) - \tan\theta}{1 + \tan\left(\frac{\pi}{4}\right).\tan\theta}\right\} d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right) d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan\theta}\right) d\theta$$

$$\Rightarrow I = \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1 + \tan\theta)\}\, d\theta$$

$$\Rightarrow I = \ln 2 \int_0^{\frac{\pi}{4}} d\theta - \int_0^{\frac{\pi}{4}} \ln(1 + \tan\theta)\, d\theta$$

$$\Rightarrow I = \ln 2\ .[\theta]_0^{\frac{\pi}{4}} - I; \text{ From (1)}$$

$$\Rightarrow 2I = \ln 2\,.\left(\frac{\pi}{4} - 0\right)$$

$$\Rightarrow 2I = \frac{\pi}{4} \ln 2$$

$$\Rightarrow I = \frac{\pi}{8} \ln 2$$

$$\therefore \int_0^1 \frac{\ln(1+x)}{1+x^2}\, dx = \frac{\pi}{8} \ln 2.$$

Exercise:

Evaluate the following integrals:

1) $\int_0^{\frac{\pi}{2}} \frac{(sinx)^{2025}}{(cosx)^{2025}+(sinx)^{2025}} dx$

2) $\int_0^{\frac{\pi}{2}} \frac{(tanx)^{2025}}{1+(tanx)^{2025}} dx$

3) $\int_0^{\frac{\pi}{2}} ln(sinx)\, dx$

4) $\int_0^{\frac{\pi}{2}} ln(1 + cosx)\, dx$

5) $\int_0^{\frac{\pi}{4}} ln(1 + tan\theta)\, d\theta$

WEAK
15

Page:17-25

Walle's theorem:

If n is a positive integer then

$$\int_0^{\frac{\pi}{2}} \sin^n x \; dx = \int_0^{\frac{\pi}{2}} \cos^n x \; dx$$

$$= \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots\cdots\cdots \dfrac{5}{3} \cdot \dfrac{3}{2} \cdot \dfrac{\pi}{2} & \text{when n is even} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdot \dfrac{n-5}{n-4} \cdots\cdots\cdots\cdots \dfrac{6}{7} \cdot \dfrac{4}{5} \cdot \dfrac{2}{3} \cdot 1 & \text{when n is odd} \end{cases}$$

Example: Evaluate by walle's formula

i. $\int_0^{\frac{\pi}{2}} \cos^7 x \; dx$

ii. $\int_0^{\frac{\pi}{2}} \cos^{10} x \; dx$

iii. $\int_0^{\pi} \sin^8 x \; dx$

Solution:

i. $\int_0^{\frac{\pi}{2}} \cos^7 x \; dx = \frac{6.4.2}{7.5.3.1} = \frac{16}{35}$

ii. $\int_0^{\frac{\pi}{2}} \cos^{10} x \; dx = \frac{9.7.5.3.1}{10.8.6.4.2} . \frac{\pi}{2} = \frac{63\pi}{512}$

iii. $\int_0^{\pi} \sin^8 x \; dx = 2\int_0^{\frac{\pi}{2}} \sin^8 x \; dx = \frac{7}{8} . \frac{5}{6} . \frac{3}{4} . \frac{\pi}{2} = \frac{35\pi}{128}$

Gamma function & Beta function

Gamma function:

Improper integral $\int_0^\infty e^{-x}x^{n-1}dx\,, \quad n > 0$ is called second Eulerian or gamma function.

It is denoted by $\Gamma(n)$, i.e., $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx\,, \quad n > 0$

Beta function:

Proper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx\,, \quad m, n > 0$ is called first Eulerian or beta function.

It is denoted by $\beta(m,n)$, i.e., $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\,, \quad m, n > 0$

Important Information:

1. $\Gamma(1) = 1$
2. $\Gamma(n+1) = n\Gamma(n)$
3. $\Gamma(n+1) = n!$; n positive integer
4. $\Gamma\left(\frac{n}{2}\right) = \frac{n-2}{2} \cdot \frac{n-4}{2} \cdot \frac{n-6}{2} \cdots\cdots\cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} . \Gamma\left(\frac{1}{2}\right)$; n odd
5. $\int_0^{\frac{\pi}{2}} sin^m x \; cos^n x \, dx = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$
6. $\beta(m,n) = \beta(n,m)$
7. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Question 1: Prove that beta function is symmetric, or prove that $\beta(m,n)=\beta(n,m)$

Proof : We know $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx \quad \cdots\cdots(1)$

Let $x=1-y$, then $y=1-x$ and $dx=-dy$.

Limits: when $x=0$, then $y=1$ and when $x=1$, then $y=0$

Therefore (1) implies

$$\beta(m,n)=\int_1^0 (1-y)^{m-1}y^{n-1}(-dy)$$

$$=-\int_1^0 y^{n-1}(1-y)^{m-1}dy$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$=\beta(n,m)$$

$\therefore\ \beta(m,n)=\beta(n,m)$ (Proved)

or, Hence beta function is symmetric. (Proved)

Question 2: Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof : From the definition of Beta function we know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx; m > 0, n > 0$$

Putting $m = n = \frac{1}{2}$ in this formula we get

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$\Rightarrow \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}dx \quad \text{since } \boldsymbol{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$$\Rightarrow \frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)} = \int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

Let, $x = sin^2\theta \;\therefore dx = 2\sin\theta\,.\cos\theta\, d\theta$

Limit: if $x = 0$ then $sin^2\theta = 0 \Rightarrow \theta = 0$

if $x = 1$ then $sin^2\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\text{So,}\{\Gamma(\tfrac{1}{2})\}^2 = \int_0^{\frac{\pi}{2}} \frac{2\sin\theta.\cos\theta}{\sin\theta\cos\theta} d\theta$$

$$\Rightarrow\{\Gamma(\tfrac{1}{2})\}^2 = \int_0^{\pi/2} d\theta$$

$$\Rightarrow\{\Gamma(\tfrac{1}{2})\}^2 = 2[\theta]_0^{\frac{\pi}{2}}$$

$$\Rightarrow\{\Gamma(\tfrac{1}{2})\}^2 = 2\left(\frac{\pi}{2} - 0\right)$$

$$\Rightarrow\{\Gamma(\tfrac{1}{2})\}^2 = \pi$$

$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ [Proved]

Question 3:

Proved that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof :

From the definition of Gamma function we know, $\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$

We put $t = x^2$ then $dt = 2x\, dx$

Limit: If $t = 0$ then $x = 0$ and if $t = \infty$ then $x = \infty$

$\therefore \Gamma(n) = \int_0^\infty (x^2)^{n-1} e^{-x^2} 2x\, dx$

or, $\Gamma(n) = 2\int_0^\infty x^{2n-1} e^{-x^2} dx$

Again putting, $n = \frac{1}{2}$ we get,

$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty x^0 e^{-x^2} dx$

or, $\sqrt{\pi} = 2\int_0^\infty e^{-x^2} dx$

$\therefore \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ [Proved]

Question 3: Evaluate

(a). $\int_0^{\pi/2} sin^6x\ dx$

(b). $\int_0^{\pi/2} sin^5\theta\ cos^4\theta\ dx$

(c). $\int_0^{\pi/2} sin^6\theta\ cos^5\theta\ dx$

(d). $\int_0^1 x^6\sqrt{1-x^2}\ dx$

Solution:

(a).

We put, I = $\int_0^{\pi/2} sin^6x\ dx$

We know that $\int_0^{\frac{\pi}{2}} sin^mx\ \ cos^nx\ dx = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

Here m=6, n=0

$$\therefore I\ = \frac{\Gamma(\frac{6+1}{2})\Gamma(\frac{0+1}{2})}{2\Gamma(\frac{6+2}{2})} = \frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(4)}$$

$$\Rightarrow I \;=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\Gamma(\frac{1}{2})}{2.3.2.1}=\frac{5\sqrt{\pi}\sqrt{\pi}}{32}$$

$$\therefore\; \int_0^{\pi/2} sin^6 x\; dx \;= \frac{5\pi}{32}.$$

(b).

We Put, I = $\int_0^{\pi/2} sin^5\theta\; cos^4\theta\; d\theta$

We know that $\int_0^{\frac{\pi}{2}} sin^m x\;\; cos^n x\; dx = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

Here m=5, n=4

$$I \;= \frac{\Gamma(\frac{5+1}{2})\Gamma(\frac{4+1}{2})}{2\Gamma(\frac{5+4+2}{2})} = \frac{\Gamma(3)\Gamma(\frac{5}{2})}{2\Gamma(\frac{11}{2})}$$

$$\Rightarrow \text{I} = \frac{2.1.\Gamma(\frac{5}{2})}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}\Gamma(\frac{5}{2})} = \frac{8}{315}$$

$$\therefore.\; \int_0^{\pi/2} sin^5\theta\; cos^4\theta\; dx \;= \frac{8}{315}$$

(c).

We Put, $I = \int_0^{\pi/2} sin^6\theta\, cos^5\theta\, d\theta$

We know that $\int_0^{\frac{\pi}{2}} sin^m x\ \ cos^n x\, dx = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

Here m=6, n=5

$$I = \frac{\Gamma(\frac{6+1}{2})\Gamma(\frac{5+1}{2})}{2\Gamma(\frac{6+5+2}{2})} = \frac{\Gamma(\frac{7}{2})\Gamma(3)}{2\Gamma(\frac{13}{2})}$$

$$\Rightarrow I = \frac{\Gamma(\frac{7}{2}).2.1}{2.\frac{11}{2}.\frac{9}{2}.\frac{7}{2}\Gamma(\frac{7}{2})} = \frac{8}{693}$$

$$\therefore \int_0^{\pi/2} sin^6\theta\, cos^5\theta\, dx = \frac{8}{693}$$

(d).

We Put, $I = \int_0^1 x^6\sqrt{1-x^2}\,dx$

Let $x = \sin\theta$ then $dx = \cos\theta d\theta$

Limit:

if $x = 0$ then $0 = \sin\theta \Rightarrow \theta = 0$

And if $x = 1$ then $1 = \sin\theta \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \sin^6\theta\sqrt{1-\sin^2\theta}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^6\theta \cos^2\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{6+2+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(5)}$$

$$= \frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2.4.3.2.1}$$

$$= \frac{5\pi}{256}$$

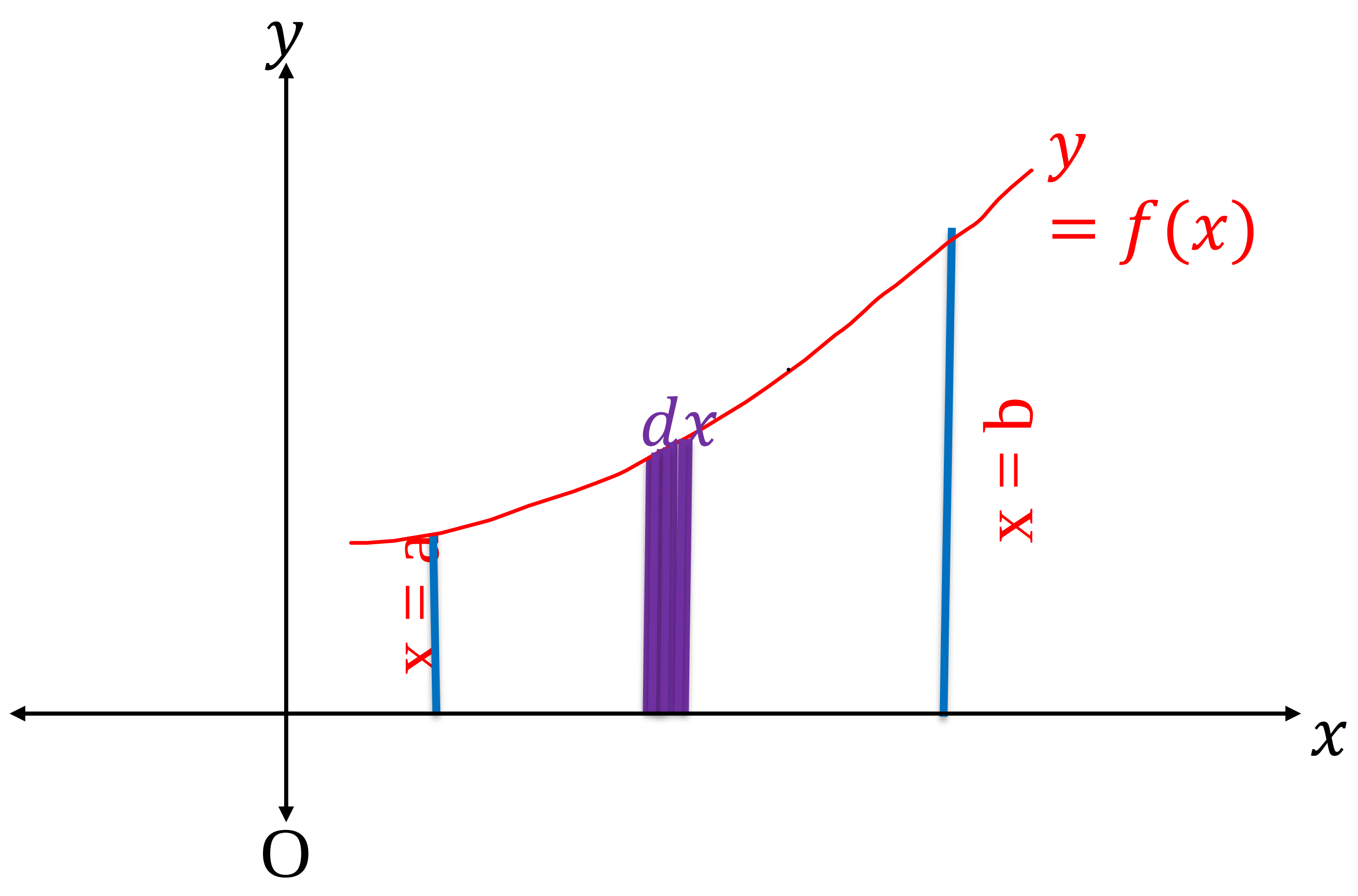
$$\therefore \int_0^{\frac{\pi}{2}} \sin^6\theta\sqrt{1-\sin^2\theta}\cos\theta d\theta = \frac{5\pi}{256}$$

WEAK
16

Page:17-25

Determining the area of the area enclosed by a curve in Cartesian coordinates

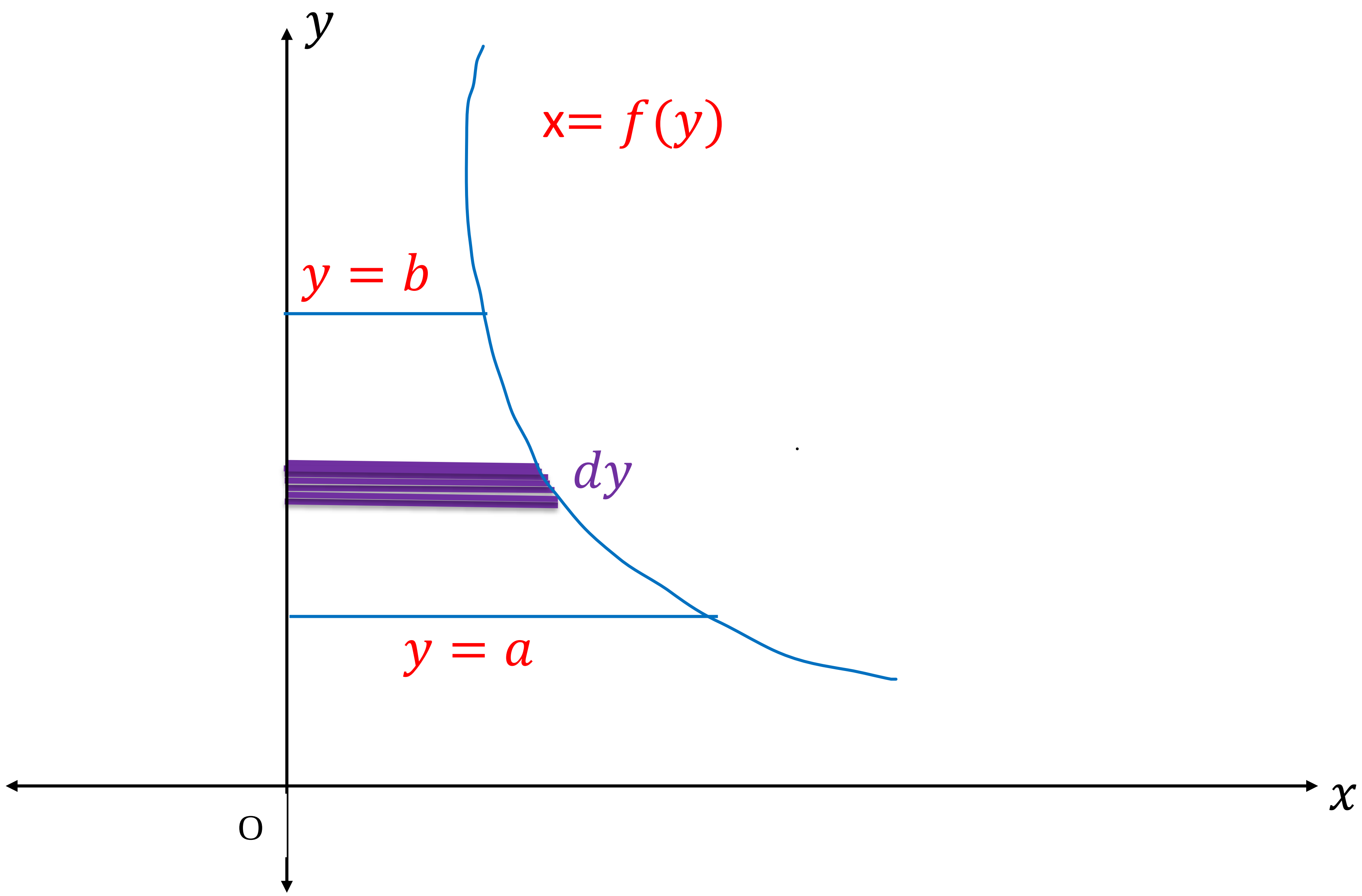
Area bounded by $y = f(x)$ & x −axis within a specific limit:



Let the function $f(x)$ be single-valued and continuous in the interval [a, b] then the area bounded by the ordinates at $x = a, x = b$, the curve $y = f(x)$, and the x-axis is

$$A = \int_a^b y\,dx \text{ or } A = \int_a^b f(x)\,dx$$

Area bounded by x= $f(y)$ & y−axis within a specific limit:

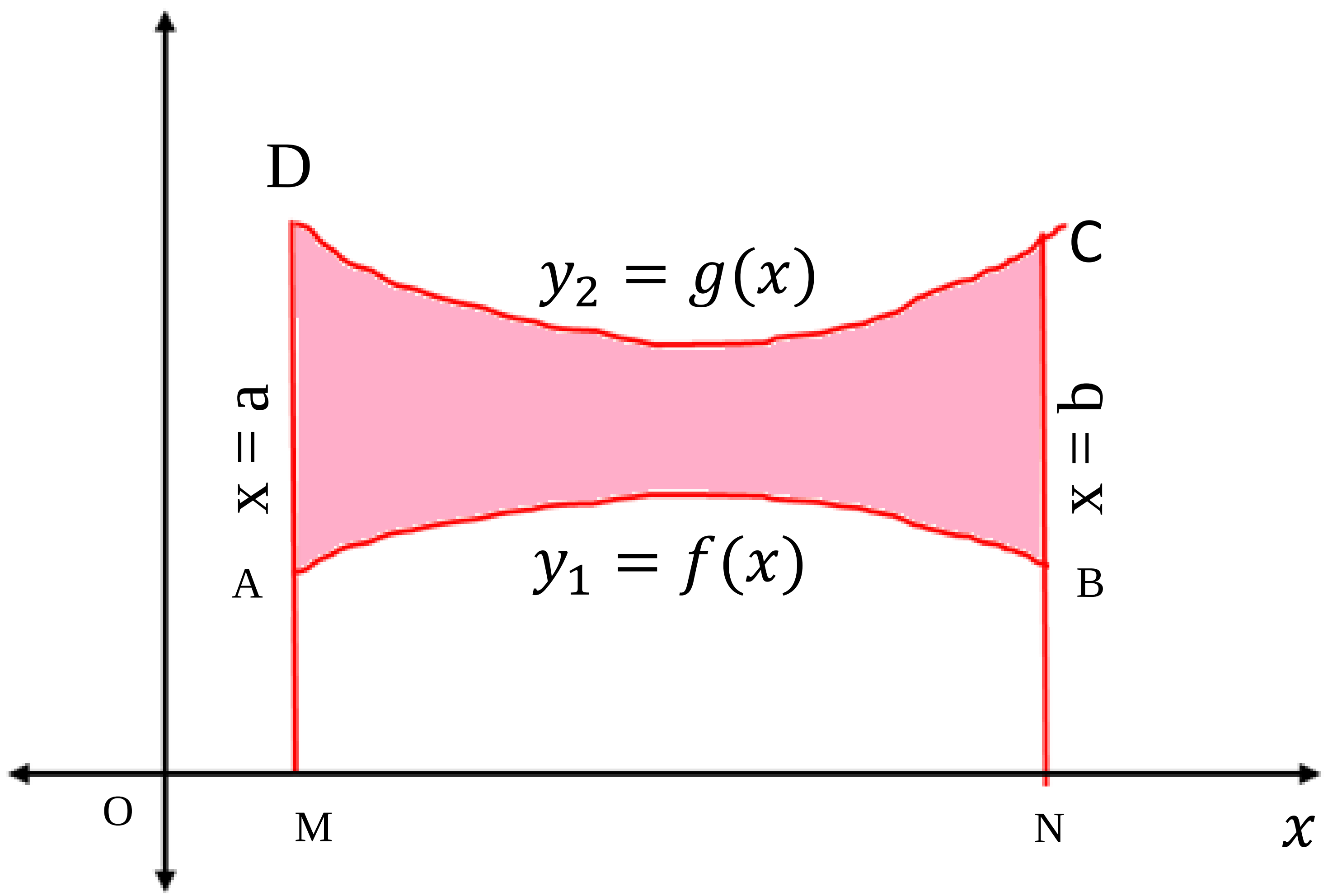


Let the function $f(y)$ be single-valued and continuous in the interval [a, b] then the area bounded by the ordinates at y= $a, y = b,$ the curve x = $f(y),$ and the y-axis is

$$A = \int_a^b x\,dy \quad \text{or} \quad A = \int_a^b f(y)\,dy$$

Area bounded by the two curves $y_1 = f(x)$ and $y_2 = g(x)$

Here we shall find the area bounded by the curves $y_1 = f(x), y_2 = g(x)$ and the ordinates at $x = a, x = b$, that is the area of ABCD.



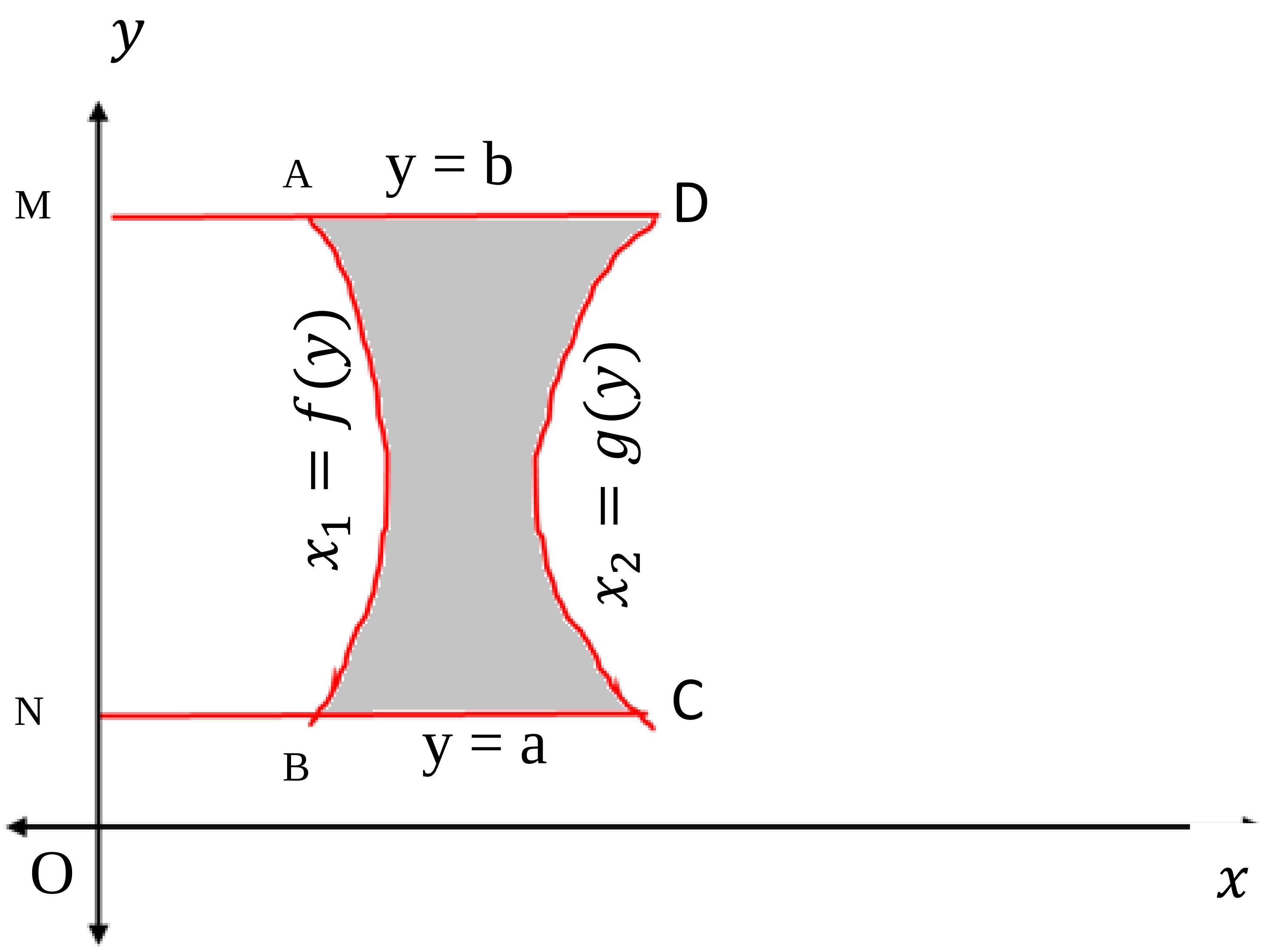
Area ABCD = Area MNCD – Area MNBA

$$= \int_a^b g(x)dx - \int_a^b f(x)dx$$

$$= \int_a^b [g(x) - f(x)]\, dx$$

Area bounded by the two curves $x_1 = f(y)$ and $x_2 = g(y)$

Here we shall find the area bounded by the curves $x_1 = f(y), x_2 = g(y)$ and the ordinates at $y = a, y = b$, that is the area of ABCD.

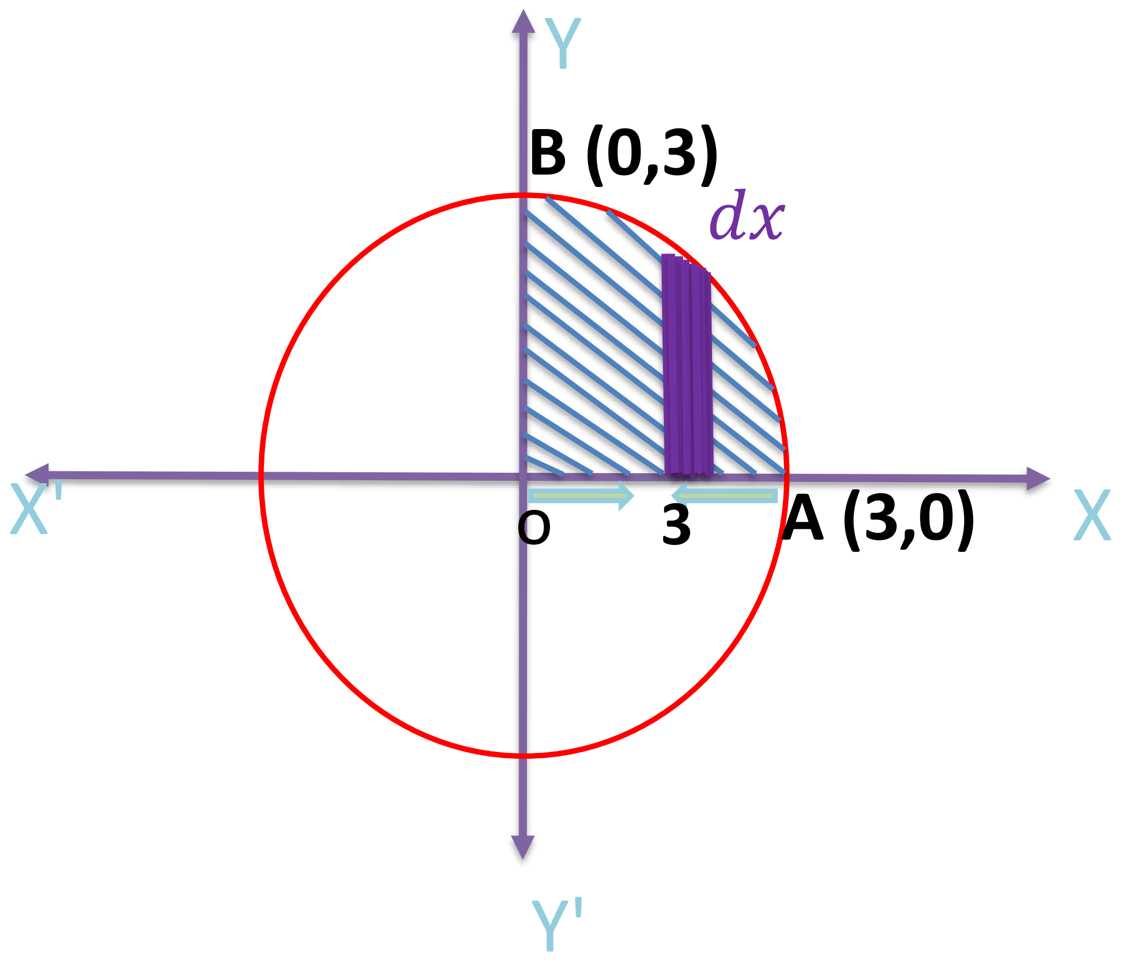


Area ABCD = Area MNCD – Area MNBA

$$= \int_a^b g(y)dx - \int_a^b f(y)dx$$

$$= \int_a^b [g(y) - f(y)]\, dx$$

Example 1: Find the area of the circle $x^2 + y^2 = 9$



Solution:

The equation of a circle is $x^2 + y^2 = 9^2$ which can be written as

$$y^2 = 9^2 - x^2 \quad \therefore y = \sqrt{9^2 - x^2}$$

The region is divided into four parts by x-axis and y-axis. If we find the area of the first quadrant and then multiply by 4 we get the total area of the circle. For the first quadrant the limit of x varies from 0 to 3. The area of the shaded region is ydx

The total area of the circle, $A = 4\int_0^3 \sqrt{3^2 - x^2}\ dx$

$$= 4\left[\frac{x\sqrt{3^2-x^2}}{2} + \frac{3^2}{2} sin^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$= 4\left[0 + \frac{3^2}{2} sin^{-1}(1) - (0-0)\right]$$

$$= 4 \times \frac{3^2}{2} \times \frac{\pi}{2} = 9\pi \text{ sq. unit}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$$

Example 2:

Show that the area of a region bounded by the parabola $y^2 = 4x$ and the straight line $y = 2x - 4$ is 9 square unit.

Solution:

Given the equations are,

$$y^2 = 4x ....(1)$$

and, $y = 2x - 4$ ....(2).

Here the curve (1) is symmetrical about x-axis. In (1) its vertex is $O(0,0)$

Now for the points of intersection, from (2) putting the value of y in (1) we get,

$(2x - 4)^2 = 4x$

$\Longrightarrow 4x^2 - 20x + 16 = 0$

$\Longrightarrow x^2 - 5x + 4 = 0$

$\Longrightarrow (x - 1)(x - 4) = 0$

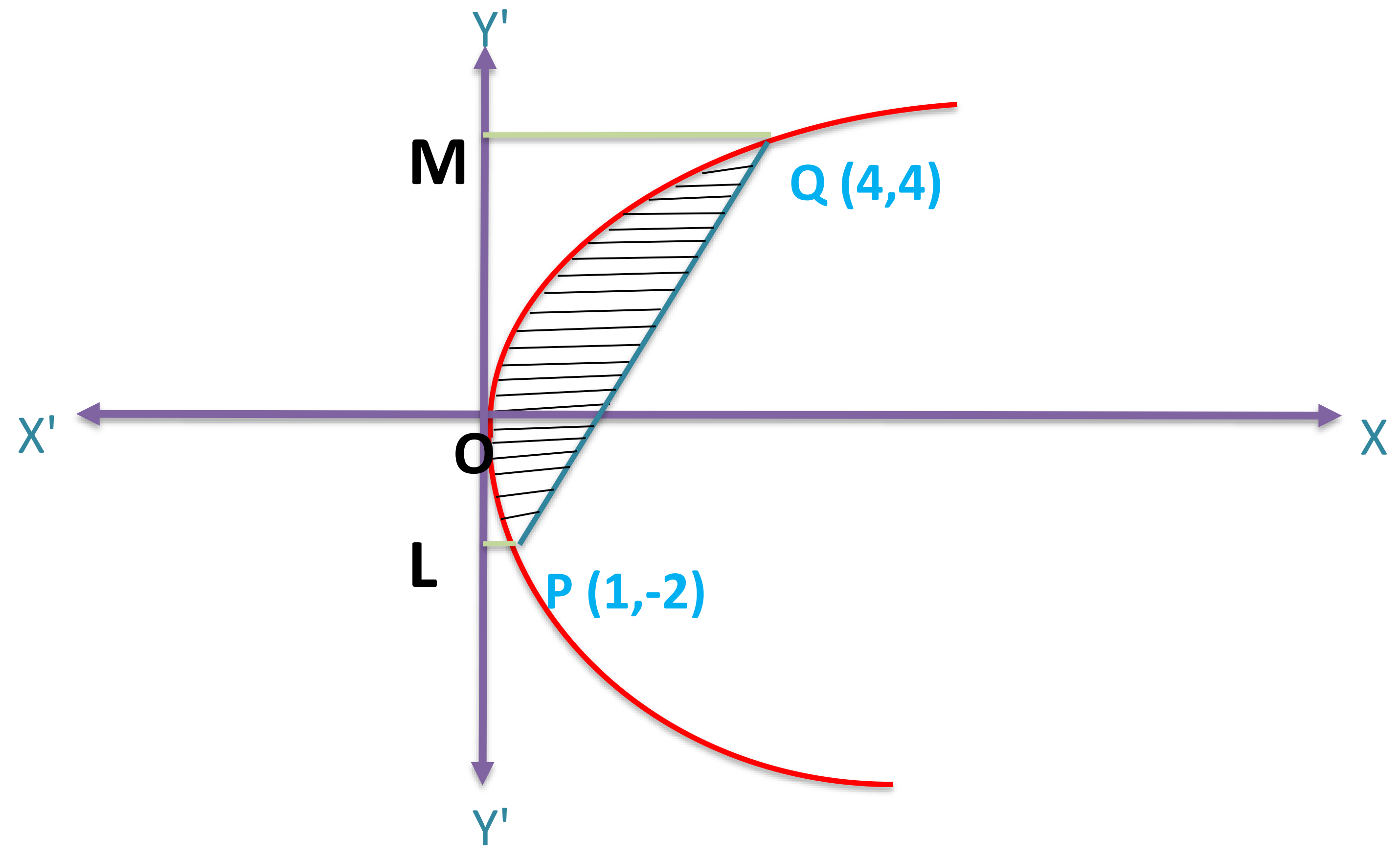
$\Longrightarrow x = 1,4$

If $x = 1$ then $y = -2$. Again if $x = 4$ then $y = 4$

Hence (1) and (2) intersect at the points $P(1, -2)$ and $Q(4, 4)$

From (1) and (2) let $x_1 = \frac{1}{2}(y+4)$ and $x_2 = \frac{y^2}{4}$

If A is the required area then $A =$ Area of PLMQ



$$= \int_{-2}^{4} [x_1 - x_2] dy$$

$$= \int_{-2}^{4} \left[\frac{1}{2}(y+4) - \frac{y^2}{4}\right] dy$$

$$= \left[\frac{1}{2}\left(\frac{y^2}{2} + 4y\right) - \frac{y^3}{4\times 3}\right]_{-2}^{4}$$

$$= \left\{\frac{1}{2}\left(\frac{4^2}{2} + 4\times 4\right) - \frac{4^3}{4\times 3}\right\} - \left\{\frac{1}{2}\left(\frac{(-2)^2}{2} + 4\times(-2)\right) - \frac{(-2)^3}{4\times 3}\right\}$$

$$= \left(12 - \frac{16}{3}\right) - \left(-3 + \frac{2}{3}\right)$$

$= 9$ sq. unit

Example 3:

Find the area bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$

Solution:

Given equations are,

$$y^2 = 4ax \text{ ………..... } (1)$$

$$x^2 = 4ay \text{ ………. } (2)$$

Here the curve (1) is symmetrical about the x-axis and the curve (2) is symmetrical about the y-axis.

For the points of intersection, from (2) putting the value of y in (1) we get,

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow \frac{x^4}{16a^2} = 4ax$$

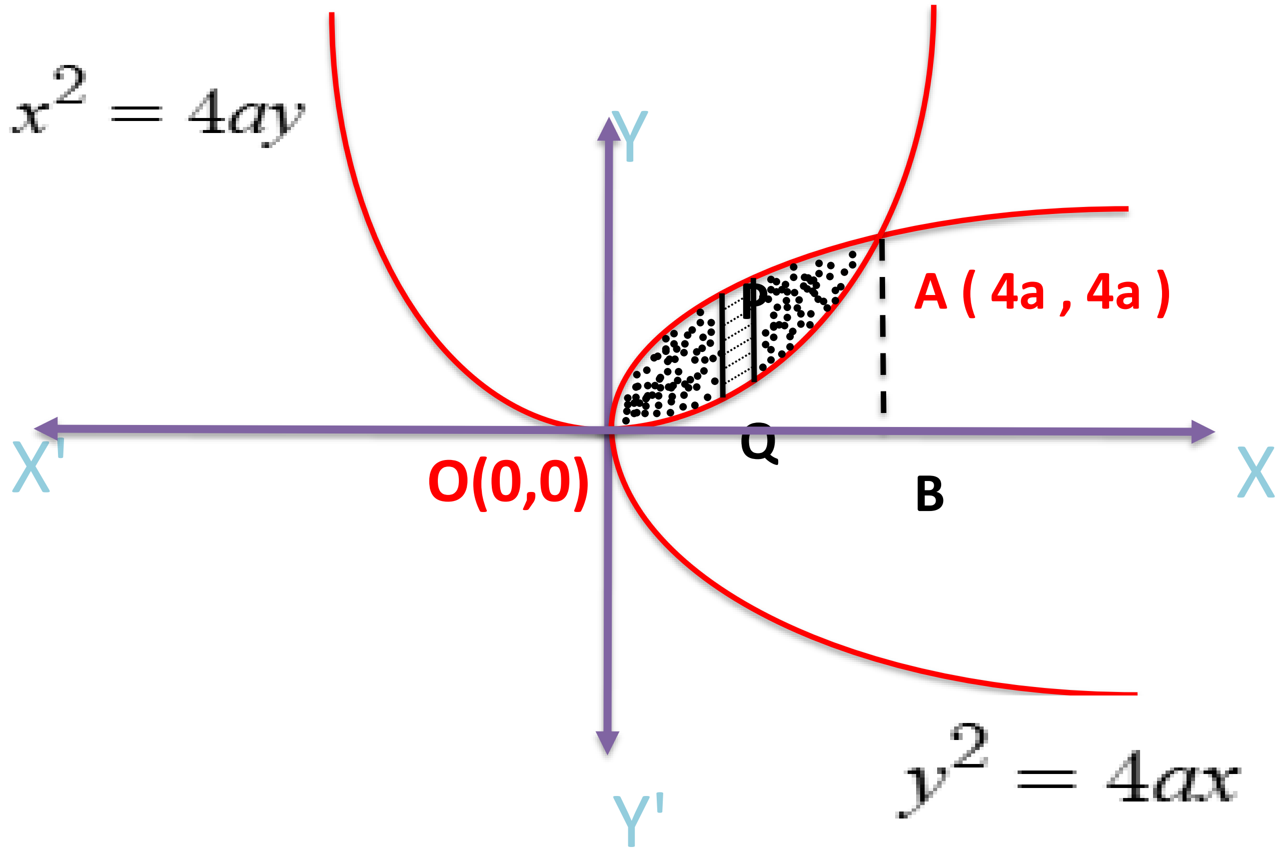
$\Rightarrow x^4 = 64a^3x$

$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x(x - 4a)(x^2 + 4ax + 16a^2) = 0$

$\Rightarrow$ x = 0, 4a.

If x = 0 then y = 0 and if $x = 4a$ then $y = 4a$
Hence the two curves intersect at the points $O(0,0)$ and $A(4a, 4a)$.



Let $y_1 = \sqrt{4ax}$ and $y_2 = \frac{x^2}{4a}$

$\therefore$Area= $\int_0^{4a}(y_1 - y_2)dx$

$$= \int_0^{4a}\left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx$$

$$= \left[2\sqrt{a}\,\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a}$$

$$= \left\{2\sqrt{a}\cdot\frac{2}{3}\cdot(4a)^{\frac{3}{2}} - \frac{1}{12a}(4a)^3\right\} - (0-0)$$

$$= \frac{16a^2}{3}\ sq.unit$$

Exercise:

1) Find the area of the circles given below.

$$(i) x^2 + y^2 = 4 \qquad (ii) x^2 + y^2 = 16$$

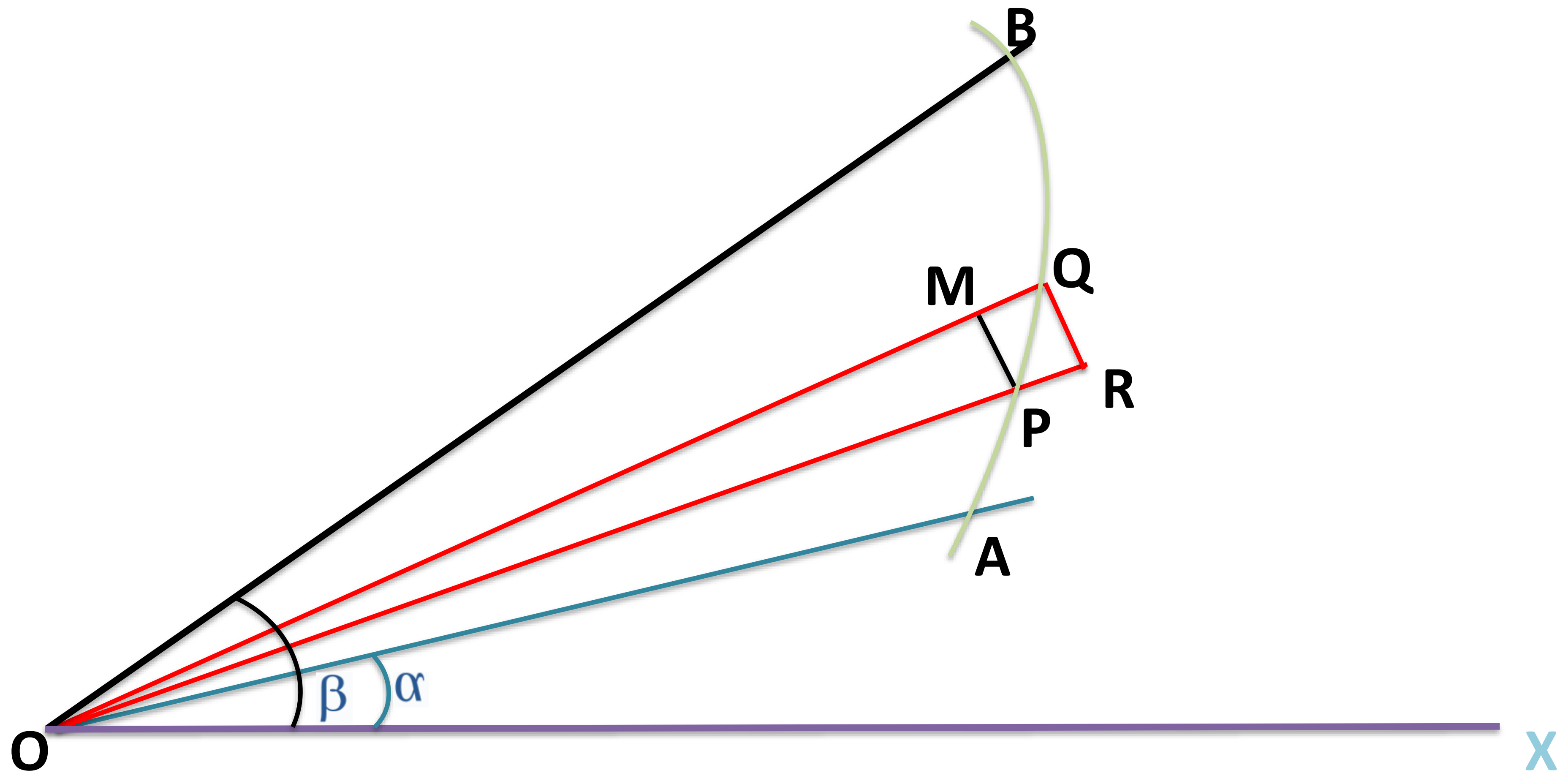
2) Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$
3) Create a problem that involves finding the area of a region bounded by two curves and solve it using definite integrals.
4) Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$ and bounded on the sides by $x = 0$ and $x = 1$.
5) Find the area of the ellipse given below.

$$(i) 4x^2 + 9y^2 = 36 \qquad (ii) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

WEAK 17

Page:17-25

Area bounded by the polar curves:



If $f(\theta)$ is a single valued continuous function of θ in the interval $[\alpha, \beta]$ and the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$ is denoted by A then $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

Example 1:

Find the area of a cardioid $r = a(1 + \cos\theta)$

Solution:

Given that $r = a(1 + \cos\theta)$

Since putting $-\theta$ for θ in (1) , the equation remains unchanged, so the curve is symmetrical about the initial line.

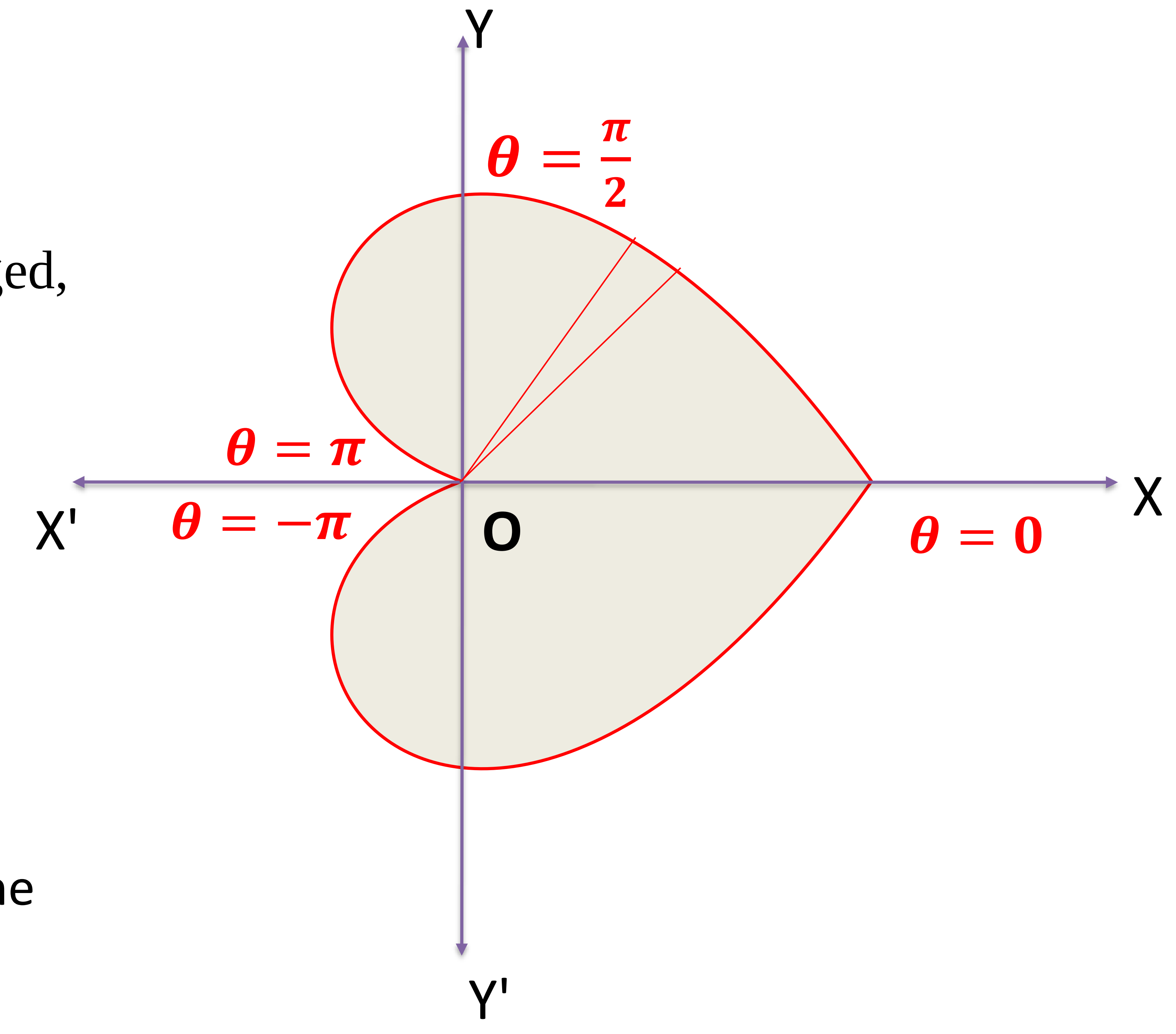
If $r = 0$ then $a(1 + \cos\theta) = 0$

$\Rightarrow 1 + \cos\theta = 0$

$\Rightarrow \cos\theta = -1 = \cos(\pm\pi)$

$\Rightarrow \theta = \pm\pi$.

Since the cardioid lies between $\theta = -\pi$ and $\theta = \pi$. So, the limits of θ are from 0 to π for the upper half of cardioid.



If A is the area of cardioid then $A = 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta$

$$= \int_0^{\pi} a^2 (1 + cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} (2cos^2 \frac{\theta}{2})^2 \, d\theta$$

$$= 4a^2 \int_0^{\pi} cos^4 \frac{\theta}{2} \, d\theta$$

We put $\frac{\theta}{2} = t$ then $\frac{1}{2} d\theta = dt \Rightarrow d\theta = 2\,dt$

Limits: If $\theta = 0$ then $t = 0$ and If $\theta = \pi$ then $t = \frac{\pi}{2}$

$$\therefore A = 4a^2 . 2 \int_0^{\frac{\pi}{2}} cos^4 t \, dt$$

$$= 8a^2 \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2}$$ [using walle's theorem]

$$= \frac{3}{2} \pi a^2 \text{ sq. unit}$$

Example 2:

Find the area of the cardioid $\mathbf{r = 1 - \sin\theta}$

Solution:

The given equation is $r = 1 - sin\,\theta$ $\cdots\cdots$ (1)

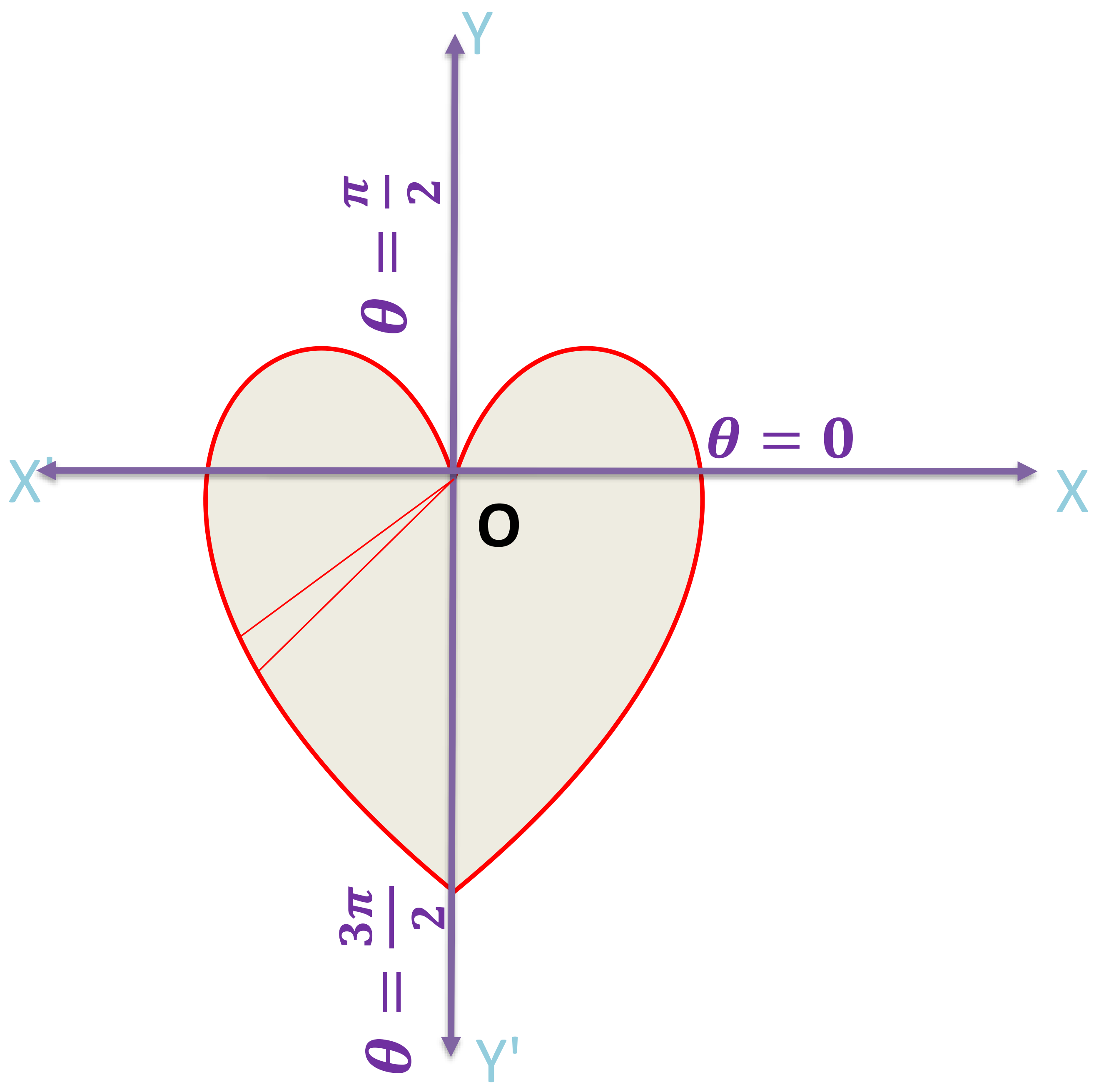
The region is divided into two parts. So, the region is symmetric about y-axis.

Putting $r = 0$ in (1) we get $1 - sin\,\theta = 0$ $\Rightarrow sin\,\theta = 1$

$\therefore\ \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Therefore, the cardioid situated between the line $\theta = \frac{\pi}{2}$ and $\theta = \frac{5\pi}{2}$.

So, the left half of the cardioid situated between the line $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$



The total area of the cardioid is $A = 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{2} r^2 d\theta$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - \sin\theta)^2 d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - 2\sin\theta + \sin^2\theta) d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(1 - 2\sin\theta + \frac{1}{2} 2\sin^2\theta\right) d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left\{1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\right\} d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(\frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \left[\frac{3}{2} \times \frac{3\pi}{2} + 0 - 0 \ - \frac{3}{2} \times \frac{\pi}{2} - 0 - 0\right]$$

$$= \frac{3\pi}{2} \ sq.unit$$

Exercise:

1. Find the area of a cardioid $r = a(1 - cos\theta)$
2. Find the area of the cardioid $r = a(1 + sin\,\theta)$.
3. Find the area of a cardioid $r = 1 + cos\theta$
4. Find the area of a loop of the curve $r^2 = a^2 \cos 2\theta$. Also find the area of all loops of the curve

THE END

THANK YOU

FOR YOU ATTENTION